Chapter Four

A Conceptualist Account of Mathematics: An Alternative to Realism and Nominalism

Abstract: Philosophers understand the discipline of mathematics to be concerned with the development and systematic investigation of formal 'abstract structures' (or ‘objects’). A major concern for philosophers is how can mathematicians have epistemic access to these abstract structures? Is mathematics about a real and objective order of abstract mind-independent entities or are mathematical objects (or ‘concepts’) mind-dependent? In previous chapters a mind-dependent response, including a theory of definition and concepts was advocated. In this chapter, the debate between mathematical 'realists' and mathematical 'nominalists' is examined. It is argued that both of these positions are false, and that another position, historically named 'conceptualism' is the best explanation of how we can have mathematical knowledge. Conceptualism is consistent with physicalism, giving it an important advantage in comparison to realism and nominalism. Conceptualism resolves pertinent epistemological and ontological issues in logic with greater clarity and plausibility than the alternatives. Before discussing the status of mathematical objects, we will need to summarize some of the historical issues in metaphysics, as well as introduce the realist, nominalist, and conceptualist positions.

I. Introduction

In chapter seven (vol. 1), we responded to the following questions:

(1) What is the *epistemic status* of axioms, definitions, and inference rules in mathematics? Can we *know* them to be *true*?

(2) Do mathematical entities (e.g., squares, numbers, and ratios) *exist*? And if so, in what sense (and how) do they exist?

(3) What is the source of mathematical truth?

In this chapter, we will again consider questions (2) and (3) as they are alternatively phrased in traditional metaphysics:

1) What kinds of objects exist?

2) What is mathematics about?

I maintain that both of these latter metaphysical questions are ill-conceived questions. This chapter responds to these historical issues: (a) the debate about the existence (or non-existence) of mathematical objects, (b) the epistemology and semantics of mathematical ‘aboutness’ and (c) the applied usefulness of mathematics. Let's summarize the three theories; realism, nominalism, and conceptualism, that address these issues:

Mathematical realism: Has three basic principles-- (1) There exist mathematical objects, (2) mathematical objects are abstract, and (3) mathematical objects are independent of persons, including their thought, language, and practices. Mathematics is about a realm of objective ‘abstract objects.’ These ‘abstract objects’ are non-spatiotemporal, nonphysical, unchanging, and causally inert. Plato is the early originator of realism, believing that mathematical objects (e.g., squares, numbers) are ‘universals’ (or Forms) and that they are eternal and cannot be created, destroyed, or changed.

Nominalism (anti-realism): Against realism-- (1) There are no 'mathematical objects' nor ‘abstract objects’ that exist independent of the human mind. Asks: (2) How can *something* that lacks a spatiotemporal address *exist*? (3) How could anyone *know* anything about a non-spatiotemporal, non-physical, unchanging, and causally inert object? An ‘abstract object’ *cannot* be what mathematics is about because there are no such objects. Whatever mathematics is about, it must be interpreted or explained without reference to existents such as abstract objects.

Conceptualism (anti-realism): There do *exist* certain kinds of objects (e.g., mathematical entities, fictional entities, propositions, possible worlds) but they do *not* exist independently of us. Abstract objects are the product of human activity. *Abstract objects* should be conceived as *mental concepts* physically located within human brains. In other words, so-called ‘abstract objects’ should be redefined as ideas, concepts, or definitions in our head where abstract ‘*objects’* are understood as physically instantiated *‘concepts.’* 'Mathematical concepts' *exist* as stipulated entities postulated in formal mathematical systems, where their 'definition' is often identified with a precise fixed (unambiguous) definiens.[1]

A 'conceptualist anti-realist' position is endorsed here. It is the hypothesis that mathematical entities exist, but that their existence is mind-dependent upon the existence

[1] ‘Mathematical formalism' as an epistemology advocated in chapter seven (vol. 1), is an epistemological companion to ontological 'conceptualism.' Formalism holds that deduced mathematical ‘truths’ are the logical consequence of a consistent set of manipulation rules in a formal system. Reasoning proceeds based upon syntactically marked regularities of expressions without an immediate concern for semantics. The content of mathematics is exhausted by the rules operating within its language. The addition of a stipulated semantics follows the rules of an assumed consistent deductive system.

of mathematicians (and anyone using mathematical concepts). For the conceptualist, mathematical entities don't exist as independent objects, but instead exist as defined 'concepts,' physically located within mental systems. Individual mathematical concepts (e.g., triangle, number, +, -, v, &) can be precisely defined as fixed definiens concepts.[2] Contrary to realism, there are no abstract objects that exist in a non-spatiotemporal realm. Contrary to nominalism, so-called abstract objects exist as concepts within the brains of humans, and there is no need to reinterpret what mathematical terms are 'about.' Again, as has been emphasized, linguistic terms (mathematical or otherwise) when used in a context do not literally refer to (nor are 'about') things.

Having in mind this sketch of the three alternative theses (realism, nominalism, conceptualism) about the ontology of 'mathematical objects' and 'mathematical concepts' we will proceed to examine the history and current status of this debate.[3]

II. Realism and Nominalism about Physical and Mathematical Objects

The debate between 'realism' and 'anti-realism' is deeply rooted in questions that were asked in ancient and medieval philosophy. The question of 'what objects exist?' is fundamental to both the ancients and contemporary metaphysicians. Metaphysics is conceived as an investigation into the most fundamental and general structures and features of reality. It attempts to analyze the most general categories of objects (known as ontic categories) as the objects that should be assumed to exist and thus constitute the

[2] Details about 'fixed definiens' concepts that is reiterated from other chapters: The term 'fixed definiens concept' (i.e., 'closed concept,' 'formal concept') is a kind of concept that has two characteristics that make up its uniqueness. First, a fixed definiens concept is a term that is stipulatively defined to *unequivocally identify* any item(s) that fall under its definition. The definiens is precise enough to distinctly exclude any entity that doesn't fall under the definition. Second, a fixed definiens concept is stable and not subject to alteration (without creating a new concept). The definiens determines what a term's proper referents (or extensions) are, if any. Fixed definiens definitions when used in mathematics typically involve an assertion of 'measurement' in a broad sense. These measurement terms will have a *necessary and sufficient conditions* definition for their proper use because, for the most part, they have been deliberately formulated that way. With fixed-definiens concepts, *the consistency of informative fixed definiens concepts* and *their relations* are sought. Although fixed definiens concepts are stipulatively defined to unequivocally identify any item(s) that fall under its definition, sometimes the definiens for a fixed definiens concept is difficult to consistently state, and may require refinement (or research), so as to be informative and consistent with other related concepts (e.g., 'derivative' in mathematics is fixed definiens but was hard to precisely specify).

[3] The following sections (II, III, IV) are long, historical, and technical. For readers wishing to understand conceptualism without its history, please proceed to section V: The Conceptualist Worldview Explained.

structure of the world. Metaphysics is about existence, identity, change, substance, matter, form, universals. actuality, potentiality, causation, and time. Most metaphysicians believe that they are not concerned with the representations of language, but with what is 'reality.'

The Realism-Nominalism Debate

Cynthia MacDonald (2005) understands the Realism-Nominalism dispute in the following way:

> … (it is) the problem that arises because things are alike. We often say that things that are alike are the same. However, two things, in being two things, are never the same. So, it seems that distinct things that are alike are both the same and not the same… When we say that different things are nonetheless, in some sense the same, we must, if possible, say what it is that is the same. One way of attempting to do this is to distinguish what might be called numeric sameness from qualitative sameness. Numerical sameness is explained by the theory of identity; a theory that tells us which inferences involving the identity sign ('=') are valid, or truth-preserving, and why. What, then, explains qualitative sameness? That is the question that a solution to the so-called problem of universals is supposed to answer. According to the Realist- one who believes that there are universals- what is the same is *the property or attribute*, that the different *things possess*. So, the Realist takes seriously the idea that there really is something the same (i.e., identical) when different things are truly said to be the same. Formulated in this way, the problem, to which the positing of universals is meant to be a solution, is a purely metaphysical one about how things are in the objective order, whereby 'objective' is meant at least 'mind-independent.' However, there are other ways in which this problem has been, and is, formulated… Armstrong (1978, p. xiii) says "It is asked how a general term can be applied to an indefinite multiplicity of particulars"… But since we know that part of what is involved in doing metaphysics is doing semantics, this way of formulating the problem gives some voice to the idea of a thing's being both the same and different (pp. 219-220, *italics added*).
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> According to the semantic formulation, the problem is how two or more particulars can satisfy the same predicate, how the same predicate can apply to two or more of them. This is a problem because the particulars are themselves distinct, and because, on the standard way of construing the relation between a thing and its properties, a thing has a property by exemplifying it. Together these imply that what makes the satisfaction of a predicate by one particular is wholly different from what it takes for the satisfaction of the same predicate by another. We seem unable to say what is the *same* about them that explains why the same

> predicate applies to them. According to the Realist, the solution is that distinct particulars can satisfy the same predicate if and only if each exemplifies or instantiates one and the same universal, which is tied in some way to that predicate…That there is generality in thought and language, apparently commits us to the existence of universals- apparently, since ordinary discourse about objects, kinds, properties, etc. is just that. A metaphysical theory will need, starting with this, to attempt to uncover from such discourse its real ontological commitments (pp. 221-222).[4]

Plato's Worldview

Plato is the father of mathematical realism. Although modern platonic realism is named after him, contemporary platonic theories are not strictly based upon Plato's original historical beliefs. For Plato, existence is the totality of beings, none excluded. The extension of being is everything. The concepts of being, identity, difference, motion and the rest (*Sophist* 254b-258e) are five all-encompassing concepts applying to reality as a whole. For Plato, 'being' is of a single genus, the concept is univocal, there is only one sense of being; either a thing exists, or it doesn't.

Plato held that everything is either in motion or at rest. Material objects can be in motion and are 'changeable' and 'corrupt.' Beings at rest, however, are 'untouchable' including an eternal realm of abstract objects. For example, on the Platonic view, the property of 'redness' exists independent of any red thing. Redness is an abstract object. Redness is a universal and individual red balls are particulars. Plato believed that universals exist independent of particulars in a realm of 'forms,' which is distinct from the spatiotemporal world. Ordinary red objects are said to exemplify or instantiate redness. Similarly, in nature, besides particular tigers, there is the 'property of tiger.' In aesthetics, besides beautiful things, there is the 'property of beauty.'

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[4] Critical comment: MacDonald believes that there are universals, and that their existence is required to explain how we can think and speak correctly in general terms (p. 222). A skeptical outlook is presented below. Do all 'objects' *possess* properties? Is there really *something* that is the 'same' when objects are numerically or qualitatively 'the same'? Do properties explain truths about 'object identity'? Our response is that *not all properties* of objects are a correspondence (or representation) of *how the world is*. Instead, *the properties* of objects *may be stipulatively defined*, for example, (1) group resemblance terms (chair, game), (2) fixed definiens terms (triangle, gross domestic product), and (3) the definiens of fictional entity terms (Santa Claus, Spiderman). If humans did not exist, the properties and definiens of these termed concepts would not exist. Further, it will be suggested that no general theory of qualitative identity is needed (or is likely possible) to explain the numerical identity or qualitative sameness of particulars.

Aristotle's Worldview About Universals and Being

Aristotle didn't believe in an independent world of forms. Instead, universals exist in particulars that instantiate them. Universals can be 'abstracted' from the entities that possess them. Universals occupy space and time, since they are 'in' things, and not 'outside' of things. For Aristotle, 'being' can mean different things— i.e., that there are different ways of being, and the primary goal of ontology is to identify those ways of being. Aristotle believed that there exists a natural hierarchy of existents with 'substances' (similar to natural kinds) as the fundamental unit of being. Aristotle was interested in clarifying a hierarchy of fundamental existents, rather than inquiring 'what exists?' As Jonathan Schaffer (2009) notes, in Aristotle's *Metaphysics,* "there is virtually no existence questions posed… the question is not *whether* numbers exist, but *how*." For Aristotle, ontological questions make sense only when restricted or qualified.

Ontological categories

Simplifying a bit, an outline of the ontology (an axiomatization) of a traditional hierarchal *a priori* metaphysics is as follows:

1) Entities exist (assume a non-empty universe).
2) All entities are either 'universal' or 'particular.'
3) Universals can be divided into 'properties' or 'relations.'
4) Particulars can be divided into 'objects' or 'tropes.'
5) Objects can be divided into 'concrete' or 'abstract.'

The top-most category, the most general of them all, is that of entity or being. The second highest level is that of *particulars* and *universals*. Particulars are individuated by their spatiotemporal location (e.g., a chair). Particulars are objects that can only exemplify but are never exemplified themselves. A universal is thought to be something that can be found (wholly present) in many places (e.g., redness). Universals are distinguished into *properties* or *relations*, and particulars can be divided into *objects* or *tropes*. *Concrete objects* and *abstract objects* are likewise presented as mutually exclusive and exhaustive categories of objects. Examples of 'concrete objects' include atoms, people, buildings, goats, planets, stamps, and so on. 'Abstract objects' are defined as objects that do not exist in space or time and are therefore entirely non-physical and non-mental. Mathematical objects such as sets, geometrical figures, properties, relations, events, are thought to be examples of abstract objects. The ontological question for the philosophy of mathematics is, do abstract objects exist, and if so, how can persons have knowledge of them? Realists maintain that abstract objects exist, and nominalists deny it.

Leibniz's Worldview: Existence, Identity and Necessity

For Leibniz, like Plato, it is axiomatic that 'everything exists' means the same as 'there exists nothing which does not exist.' Assuming a non-empty universe, he reasons:

1) To exist is to be identical with something.
2) Everything is identical to itself.
3) Everything is identical to something (viz. itself).
4) Everything exists.

We can interpret the premises of this argument as consisting of implicit definitions (i.e., axioms) about the concepts of existence, identity, and thing, with a validly prescribed conclusion about what 'existence' is, in terms of the universal quantifier and things. All things are existents, and there are no nonexistent things.[5] We can restate and expand upon Leibniz's cluster of metaphysical thinking about ontology as follows:

Assume: Non-empty universe.
Something(s) exist.
To be a thing is to exist and have an identity.
Everything possesses at least one property (i.e., self-identity).
Some things have properties in common (e.g., redness).
Therefore, properties (and relations) exist and constitute the identity of everything.

Leibniz's Law of the Indiscernibility of Identicals: If x is identical with y, then x and y are indiscernible: any property of x is also a property of y, and vice versa. It is the principle that identicals have all properties in common. (In other words, $x = y$ if, and only if, x has every property which y has, and y has every property which x has).

Any identity criterion has the following form: 'If x and y are things of kind F, then $x = y$ if and only if…'. An identity criterion provides the supposed necessary and sufficient conditions for the identity of x with y. The criteria of identity (for any entity) is thought

[5] 'Everything exists' is logically equivalent to 'there exists nothing which does not exist.' Everyone should agree that 'to exist' is 'to be' and 'to be' is 'to be identical with something.' It doesn't make sense to split the totality of things into those that exist, and those that lack it.

to be determined by using *a priori* reasoning and implementing appropriate metaphysical and ontological principles. An identity criterion for existing object(s) seeks to specify the condition(s) under which *x* and *y* are one and the same thing, independently of our being to establish whether they are. It is thought by metaphysicians that we need a criterion for the identity of objects *x* and *y*, so as to determine whether we have one thing or two.

What are the conditions under which *x* and *y* are the same? Leibniz answered with the 'logical law' of the indiscernibility of identicals.[6] MacDonald (2005) states "Leibniz's Law embodies the very idea of what it is for an object to be self-identical" (p. 66).[7] The law is seen as valuable for extrapolating the laws of reflexivity, symmetry, and transitivity. Leibniz shared with Plato a desire to explain 'similarity' and 'difference' to motivate realism with respect to universals.

Frege's Worldview: Linguistic Reference, Meaning, and Compositionality

For Frege and most modern logicians, it is assumed that linguistic expressions (e.g., sentences, phrases, and words including proper names) are written marks (or sounds, or tactile expressions) that possess 'semantic values' (or 'semantic properties') whereby a linguistic expression can *mean* something when used in a context. A sentence is composed of linguistic expressions that have meanings in a well-formed sentence. One doesn't understand the meaning of a word in isolation, but only in the context of a sentence. For example, a speaker's *linguistic expression* (e.g., the proper names 'Aristotle,' 'Hesperus') can *mean* something (e.g., a person, a planet) when used in

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[6] Critical comment: Is the 'identity' of an individual physical object an independent objective property that every object has, or possesses? Or is identity (sometimes) a human projection onto things? Consider a single entity, an automobile. With the completed major repair of an automobile (e.g., installing a new engine) *we might want to say* that it is the *same* automobile, but is it really? In its biological evolution, a grape becomes raisin; but does it remain the same thing? The identity of physical entities (e.g., automobiles, raisins, the Ship of Theseus, individual persons) oftentimes depends on persons' implicit (or explicit) concepts and weighted evaluations of what counts as the same item, as well as the properties of the item. There are no objective independent material conditions that determine what it is for a physical thing (excluding natural kinds) to persist through time and change.

[7] As stated in chapter 3 above, Lowe (2002) says that it is an indisputable fact that 'everything is identical with itself.' Everything whatsoever stands in this relation to itself and to nothing else. It can only hold between a thing and itself (p. 23). As an *a priori* science, Lowe maintains that there is a strong case for saying when we talk about identity over time, we are genuinely talking about *identity*, that is; about the relation which, of necessity, every object bears to itself, and only to itself (p. 91).

context. The meaning of a sentence (in context) determines its sense and reference. At the level of extension, singular terms (proper names, definite descriptions, indexicals, and natural kind terms) *refer* to objects; predicates *refer* to functional concepts (e.g., 'mother' = Mx), and sentences *refer* to truth-values. Logical axioms and theorems, as well as all thoughts and their constituents are abstract objects, imperceptible to the senses, but graspable by the intellect. Frege maintained that *words do not refer* to psychological ideas (e.g., concepts, or definitions) because different people can associate different ideas with the same word. Instead, words refer to objects and functional concepts.

With respect to mathematics, Frege thought of numbers, lines, and shapes as logical objects, because they can be identified with the *extensions of concepts*. The 'extension' of a concept is the set of things falling under a concept. Frege conceived that general terms (predicates, adjectives), quantifiers (all, some, exists, etc.) and connectives (and, or, etc.) all *refer* to *functions* of various types. For Frege, a concept is identified as a function. In a sentential formula (e.g., '**x** is an elephant'), Frege viewed the objects of predicates as being 'unsaturated,' or holes that have to be 'filled in' with singular terms in order to generate thoughts. Whether '**x** is an elephant' or '**x** is a unicorn' or '**x** is prime' is satisfied (as true) depends on whether the **x** is an 'object' or 'extension' (e.g., concrete or abstract) that fulfills (or exemplifies) the concept of elephant, unicorn, or prime.

Frege's view about mathematics might be characterized as follows: The language of mathematics refers to and quantifies over non-spatial abstract mathematical objects. An object is abstract if, and only if, it is non-mental, non-physical, non-sensible, and causally inefficacious. Abstract mathematical objects exist *independent* of human language, thought, and practices.[8] The distinction between 'abstract' and 'concrete' objects is exclusive, exhaustive, and an intrinsic feature of reality. Every object falls into one of two categories: concrete or abstract.

[8] Frege thought that when formal semantics is applied to mathematics, this naturally leads to the postulation of 'mathematical objects.' If "7 +5 = 12" is true and its singular terms have reference, then there must be objects denoted by the numerals. Thus, there are abstract objects. Frege also used the notion of 'property' to claim mathematical objects exist. With the sentence 'Three is prime,' it seems that 'three' is a proper name that refers to an object, and 'prime' is a predicate that ascribes a property to this object. Since this sentence is true, it follows 'three' succeeds in referring to an object, and hence there are mathematical objects.

Tarski's Worldview: Classes, Extensions, and Identity

Alfred Tarski (1946) accepts Frege's talk about the 'extensions of concepts' where those 'extensions' or 'objects' are clearly defined and unambiguously determined. Apart from 'individual things' logic is concerned with 'classes' of things, where classes are more often referred to as 'sets.' The idea of a 'theory of classes' involves the idea of individual objects **x**, **y**, **z** being possible elements (or members) of a class. For example, the numbers 1, 2, and 3 belong to the class of integers, where fractional numbers do not. It is assumed in logic that to every sentential function containing just one free variable, say '**x**', there is exactly one corresponding class having as its elements those, and only those, things **x** satisfy the given function. Tarski explains that the concept of *'properties'* used in Leibniz's Law of the Indiscernibility of Identicals can be *eliminated* in favor of *'class'*:

> … It is frequently said of a sentential function with one free variable that it expresses a certain property of things—a property possessed by those, and only those, things which satisfy the sentential function (the sentential function 'x is divisible by 2', for example, expresses a certain property of the number x, namely divisibility by 2, or the property of being even). The class corresponding to this function contains as its elements all things possessing the given property, and no others. In this manner, it is possible to correlate a uniquely determined class with every property of things. And also, conversely with every class there is correlated a property possessed exclusively by the elements of that class, namely the property of belonging to that class. It is accordingly in the opinion of numerous logicians, it is unnecessary to distinguish at all between the two concepts of a class and of a property; in other words, a special "theory of properties" is dispensable—the theory of classes being perfectly sufficient (p. 72).

To repeat, Tarski concludes that Leibniz's law can be *reformulated* using the concept of 'class' instead of 'property.' In other words, **x** = **y** if, and only if, every class which contains any one of the things **x** and **y** as an element also contains the other as an element. With this, Tarski maintained it is possible to define the concept of identity in terms of the theory of classes.

Quine's Worldview

W.V.O. Quine (1948) rejected the idea that ontology requires an *a priori* (i.e., independent of the senses, rational) investigation into the fundamental building blocks of reality. Quine's ontological realism holds that we can determine 'what exists' by just examining the entities that are endorsed by our best scientific physical theories of the world. Quine's prescription for identifying 'what there is' was to never admit kinds of entities that we lack identity criteria for instantiating. For example, if **x** and **y** are material objects, then **x** is identical with **y** if and only if **x** has the same spatiotemporal location as **y**. Material objects are identified by their address in space-time. Quine assumed that there cannot be two numerically distinct material objects occupying the same place at the same time. With the mathematical objects found in scientific notation, their identity could be ascertained (or proved) within the language of mathematics.

Quine's proposed methodology was to translate scientific sentences into first-order logic and determine the ontological commitments of the sentences based on what those translations quantify over. Quine believed in 'ontological relativism.' There is no single, objectively correct ontological theory about the world.[9] Ontology is not a science that reveals pre-theoretical commitments, but a theoretical effort to get clear what kinds of entities we are disposed to admit by our best scientific theory of the world. In writing the ontological catalogue, it consists of what objects make true, the true sentences of our ordinary and scientific language. For Quine, the notion of 'being' is captured by the existential quantifier: to be is to be a value of a variable.[10]

Quine's indispensability argument gives a broad empirical defense of 'mathematical platonism' since mathematical entities are indispensable for making true assertions about the world. That there is a mathematical component in describing physical laws, this implies that mathematically 'abstract' objects exist. Quine was against the physicalist idea that reality is exhausted by the physical.[11]

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[9] Like Hilary Putnam, Quine admitted that there isn't a unique, objective, canonical logical form, wholly free of commitments about the nature of reality that it is used to describe. To think of the world as a domain of individuals, characterized in predications and quantified over; is to attribute a metaphysical structure.

[10] Quine's (1981, p. 72) 'naturalism' is the view that all things are physical entities, and that acceptable methods of justification and explanation (in any scientific endeavor) are continuous with those in physical science. Naturalists hold that philosophy and physical science (as well as other sciences) should establish knowledge about the natural world through *a posteriori* investigation. All concepts can be ultimately and entirely defined in terms of logic, mathematics, and physical science.

[11] But by limiting 'what exists' to what is symbolized in logic, it turned out that Quine's scientific translations didn't admit 'properties,' 'meanings,' or 'fictional characters.' Further, Chihara (1990) states that when translating Coulomb's Law, a widely accepted law in science, into a first-order sentence of logic,

Quine wasn't interested in translations outside of scientific disciplines.[12] Explicitly, Quine sought to:

1) Paraphrase our best scientific theories into canonical interpretation.

2) Take note of the ontological commitments of such paraphrase.

3) Accept these ontological commitments and just these.

But, as Francesco Berto & Matteo Plebani (2015) state, the implementation of the first step is difficult:

> … The view that we can mechanically, or almost-mechanically, (a) turn the sentences composing our theories into the canonical notation; (b) list the sentences that begin with an existential quantifier and (c) read the ontological commitments of our theories off those sentences—is an oversimplification. Theories are not easily identifiable objects. (p. 40).

Quine's reference to 'best' scientific theories also indicates a *convergence* in a discipline, and this isn't always the case, as when new *divergent* theories and entities are introduced.

III. What Exists? 'Existence' is a Group Resemblance Concept. Physicalism.

Attempts to answer, 'what exists, and what kinds of objects exist?' from a metaphysical perspective remains unsettled. But even with the rejection of realist metaphysical ontologies, the question still remains in general form, 'what exists?' An Aristotelian-like reply from a number of contemporary philosophers is endorsed here. It states that the question 'what exists' is asked and answered relative to specialized disciplines: The astronomer asks whether there exist 'black holes?' The biologist asks what kinds of 'anti-bodies' exist? Archaeologists ask whether the lost city of El Dorado

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it renders it literally false, as the law applies to ideal or hypothetical situations: "…there are particular problems connected with the nature of physical theories and laws. Even such a fundamental law as Coulomb's Law, a version of which states that every point charge produces a surrounding spherically symmetric electromagnetic field, raises serious problems: *straightforwardly translating Coulomb's Law as a first-order sentence* results in something that is clearly *false*… the idea is that what is described by Coulomb's Law are ideal situations, or situations in which complicating factors are absent… so it becomes questionable whether a translation of the law into the extensionalist's first-order language is even feasible" (p. 11, *italics added*).

[12] Quine thought that the view that there are 'many kinds of existents' was nonsense. He took this attitude as a consequence of his view that existence is expressed by existential quantification (1960, p. 242). Many ontological pluralists agree that quantification captures being or existence, but pluralists add that there is more than one way of existing, and thus there is more than one meaning for quantificational expressions.

exists? The ordinary sense of 'existence' is context-relative according to an inquirer's interests. Existence is a *group resemblance* concept. When evaluating existence claims, one normally presumes a category (or kind) of entity that is being discussed. What 'exists' is postulated by the different physical sciences,[13] mathematics (e.g., numbers, points), social sciences (e.g., economic equilibrium), fiction (e.g., characters in novels), pragmatic stipulations, etc. There is *no true metaphysics* about what *'really' exists.'*[14]

But with regard to what is the *'fundamental existent'*, it is *assumed here* that *physical (or material) entities* are the primary existent from which all other entities are composed of. *Physicalism* is the doctrine that *only material entities exist*, and that any other entities are constituted as physical entities. One definition of 'existence' is that '**x** exists if and only if **x** belongs to the space-time-causal system that is our world' (Bruce Aune, 1985, p. 35). According to Daniel Stoljar (2010), the materialist position has become "something like a consensus position within analytic philosophy in the 1960's and has remained so, or very nearly so, ever since. Philosophers such as Quine, Smart, Lewis, Armstrong, and many others are all materialists" (p. 2).

Stoljar is among physicalism's critics. The main problem with materialism-physicalism, as he sees it, is how to state a positive characterization of what a 'physical' entity is, and state physicalism (and naturalism) as non-trivially true doctrines. John Collins (2015) agrees and says that with physicalism "there is no ready way of tethering the metaphysical doctrine of physicalism to scientific inquiry" (p. 94). It is Collin's opinion that without a clear conception of what 'physical entities' are, physicalism

[13] The existents (or objects of study) of physical science are subject to causal influences (unlike mathematical entities).

[14] In opposition, Paul Bloomfield (2001) claims that there is a *true metaphysics* and what '*really exists*' *does not depend* on *our judgments about existents*: "Metaphysical realists want to talk about the world, not words. Thus, one may be a realist about scientific or moral discourse and yet non-realist about aesthetic discourse or discourse about fictional characters. A realist's position regarding a particular discourse is to stake out the claim that the items that are the subject of the discourse are real (using the vernacular); that is, they exist "out there" (with a pointed finger), externally in the world. Realists about things that are **x** say that were we to make a list of the contents of reality or the world (an ontology), the things we call "X"'s would have to be included, in the same way that sealing wax, sunshine, and succotash have to be included. These "real" items also have to exist in a fashion independent from our thinking and talking about them; for example, they cannot be "in the eye of the beholder." So, being a realist opens up the possibility that our judgments about what items exist and which do not may be mistaken, for what "really" does exist does not depend on us (our judgments) for existence" (pp. viii-ix).

shouldn't be understood as a true or false metaphysical doctrine, but a commendation for the aims and means of mature science.[15] Despite these minority protests, 'physicalism' is *endorsed* here. Physicalism is the doctrine that *only material entities exist*, and that any other existent (e.g., mathematical, fictional) has an underlying physical basis.[16] It includes the view that mental states (e.g., different kinds of concepts, beliefs, desires, values, and intentions) are *located* as physical states in the physiology of humans.[17]

IV. Nominalism Against Platonism: Do Properties Exist?

The modern doctrines of mathematical 'platonism' and 'nominalism' are contemporary positions, inspired by, but not directly based upon their historical origins. Contemporary doctrines of mathematical platonic realism, and the reality of properties, such as those endorsed by Michael Resnik (1997) and Stewart Shapiro (1997), are largely independent of original historic aspiration.[18]

As outlined above, 'universals' are traditionally divided into 'properties' or 'relations.' A 'property' is understood as a quality, aspect, feature, peculiar to an individual or a thing. Properties (i.e., attributes, qualities, features, characteristics) are predicated of things or attributed to them. Properties are 'entities' that individual things are said to bear, possess, or exemplify. Predicates are, likewise, thought to express properties and are exemplifiable. Predicate terms (e.g., wise, walks, loves) are thought to

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[15] Timothy Williamson (2014) agrees with Collins. He states that one motive behind naturalism is the aspiration to think in "a scientific spirit" and to emphasize "values like curiosity, honesty, accuracy, precision, and rigor." Williamson is concerned that naturalism (and the scientific method) can't account for mathematics and worries that naturalism is "an obscure article of faith" (pp. 30-31).

[16] Consciousness can be explained by physics, biology, neuroscience, and cognitive science.

[17] The neuroscientist and philosopher Sam Harris (2010) states that "the strict localization of any mental state is difficult because the human brain is characterized by massive interconnectivity; it mostly talking to itself. And the information it stores must also be more fine-grained than the concepts, symbols, objects or states that we subjectively experience. Representation results from a pattern of activity across networks of neurons and does not generally entail stable, one-to-one mappings of things/events of the world or concepts in the mind, to discrete structures in the brain." (pp. 119-120).

[18] Resnik and Shapiro endorse platonistic structuralism. Mathematical theories provide true descriptions of mathematical structures, which are abstract. We acquire knowledge of mathematic structures by constructing axiom systems, which provide implicit definitions of structures. Mathematical objects and patterns exist independently of us and our constructions. Mathematic truths obtain independently of our beliefs, theories, proofs. The mathematical realm is independent of our mental life.

be true of things and thus denote a property or relationship. Relations, (e.g., loving, between, tall) are also predicable and exemplifiable, on par with properties.

'Nominalism' is defined as the view that there are no universals (nor abstract objects). Given that nominalists deny 'all entities are either universal or particular,' this is a rejection of the second highest hierarchal axiom of the 'ontological' categories in traditional metaphysics above (i.e., all entities are either 'universal' or 'particular'). Nominalists believe that particulars (or individuals) are the most fundamental category of being, and that universals (or other abstract objects) may be admitted into our language, only if they can be reduced to particulars. Nominalism claims that there are *no universal existents* such as 'properties' and 'relations.' For example, if it is true that 'John smokes,' the world contains is John, but not another thing, the property of smoking. All that exists is a 'smoking John.'

Also, nominalists criticize the talk of properties as redundant and misleading. Do 'properties' really *exist* as entities that make up the *composition* of an entity (e.g., physical or mathematical entity)? For example, a standard objection is that we often say that 'an apple is red,' and not that 'an apple has *the property* of redness' even though these two sentences express the same proposition. When one says, 'some dogs are *white*,' one commits oneself to there being dogs and white things, but *not to* the existence of the properties of dogness or whiteness. Also, the Eifel Tower and Empire State Building have something in common as being tall. But being 'tall' doesn't imply the buildings *possess* a property (or relation) of 'tallness.'

Mathematical Nominalism: The Twentieth Century

Modern mathematical nominalism began with Nelson Goodman and WVO Quine in a journal article, "Steps Toward a Constructive Nominalism" (1947) which was widely considered unsuccessful. But instead of just denying the existence of abstract objects, Goodman and Quine sought a positive reconstructive project that reconstrued or reinterpreted mathematics, so as to eliminate reference to the existence of non-spatiotemporal, nonphysical, unchanging, and causally inert 'abstract' objects. While their project didn't work, other nominalists have followed. How does mathematics work, these authors ask, if there are no nonphysical abstract objects?

Nominalism, as a positive reconstruction project, is fundamentally concerned with the status of 'existence' quantifiers as found in first order logic. Nominalism either (a) denies that mathematical objects, relations, and structures exist at all, or else (b), mathematical objects don't exist as abstract objects.[19] Nominalism's task, as understood, is to show how to interpret mathematics without commitment to the existence of mathematical objects. A nominalist needs to specify some suitable replacement theory for what have been labeled 'abstract mathematical objects.' It is believed by some nominalists that physical scientific theories, now using numbers, can be (hypothetically) *rewritten* to *eliminate reference* to numbers. Other nominalists seek to eliminate reference to abstract objects, in favor of statements about the physical world. Nominalist theories deny the existence of abstract objects, by postulating lengthy formal vocabulary replacements, often while favoring a naturalist epistemic worldview.[20]

The clearest and most authoritative book on the topic of mathematical nominalism is *A Subject with No Object* by John Burgess and Gideon Rosen (1997). They make clear that the *platonism-nominalism debate* is *distinctively a debate among philosophers*. Professional mathematicians are generally satisfied that axiomatic set theory provides a unified, general framework for mathematics, conforming to their ideal of rigor. The nominalist, however, finds difficulty in understanding how 'abstract objects' are supposed to exist. We generally understand how we can possess (empirical) knowledge of concrete objects, but knowledge of abstract objects is mysterious. *Nominalists seek an alternative account to 'abstract objects' because of epistemic concerns.*

Burgess and Rosen admit that they possess broadly realistic-platonist intuitions. They understand the nominalist's problem as simultaneously preserving mathematics as a

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[19] Critical comment: Nominalist theories are unduly fixated on 'existence' claims, and with formally reinterpreting the existential quantifier in first-order logic (e.g., such as Charles Chihara, 1990, 2005).

[20] A version of nominalism is 'mathematical fictionalism' that is endorsed by Harty Field (1980), Mark Balaguer (1998) and Mary Leng (2010). It is an 'error theory' about mathematical discourse. It states that, although we normally take many mathematical statements to be true, we do so 'mistakenly.' Shapiro (2016) states that "For the fictionalist, the pursuit of mathematics involves tracking the consequences of the 'stories' we tell about the fictional objects called 'natural numbers,' 'complex valued functions,' and the like. If knowledge of what follows from what is tractable, then, for the fictionalist, so is the non-trivial pursuit of mathematics—what passes for mathematical knowledge on that view" (p. 626). Other nominalist theories include modal structuralism (Geoffrey Hellman, 1989) and deflationary nominalism (Jody Azzouni, 2004).

distinctive subject, while abolishing its distinctive objects. For Burgess and Rosen mathematical realism is an intuitive world view. Intuitively, we believe that 'numbers exist' and that they are not supposed to be like concrete things like rocks, trees, or people. We acquiesce, not only in their existence, but also in their abstractness. The authors state that revisionary nominalism must provide positive motivation and reasons for abandoning abstract object existence claims. They ask why the nominalist, who disbelieves in abstract entities, seeks to reconstrue theories that involve them. Is such a project necessary? Although Burgess and Rosen sympathize with a *prima facie* case for denying the existence of abstract objects, they believe that *unless* the nominalist can provide a tenable reconstruction or reconstrual of mathematics, which seems difficult; their technical, formalized theories of how a mathematical theory can be paraphrased into some nominalistic theory are bound to fail.[21]

The Conceptualist View of Properties

The conceptualist shares the nominalist's suspicion of 'properties.' Let us consider some so-called 'moral properties.' For example, if **S** is 'courageous,' is the predicate 'courageous' to be identified with the property of courage? Is courage a property? If **S** is 'honest,' does **S** *possess* the property of honesty? Is honesty a property?

Or is it a better more common-sense explanation to say that persons normally attribute (i.e., predicate, evaluate) honesty and courage to **S** as an entity fulfilling certain characteristics under these concepts? It seems to be a better explanation that persons evaluate **S** under the group resemblance concepts 'honesty' and 'courage' based upon a loose cluster of *disjunctive characteristics*, based upon the reportive definitions of these terms. It is a set of traits (e.g., bravery, conquering fear) that compose the definiens of 'courageous' as a predicate. 'Courageousness' is best understood as a set of similar actions or dispositions, as dictated by our interests and evaluations, rather than as something

[21] Paolo Mancosu (2008) criticizes metaphysical theories of mathematics: "I think that philosophy of mathematics has to a great extent been hijacked by metaphysics… this has had as a consequence an extremely narrow view of mathematical epistemology within mainstream philosophy of mathematics, a view partly due to the over-emphasis on ontological questions. For the most part, current epistemology of mathematics has not addressed at all matters relating to fruitfulness, understanding, explanation, and other aspects of mathematical epistemology" (p. 200).

objectively 'residing in' a person. It is a matter of **S**'s character (objectively stated) that qualifies **S** as 'courageous.' Similarly, if **S** is deemed 'honest,' then **S** is characterized as being truthful, sincere, and trustworthy. The objective characteristics of **S's** actions and character don't constitute **S** as possessing an objective *property* of 'courage'. The objective characteristics of **S's** actions and character don't constitute **S** as possessing an objective *property* of 'honesty'. The reification of 'objective properties' isn't necessary.[22]

When persons attribute the predicates of 'courageous,' 'honest,' 'tall,' 'red', or 'white,' to entities falling under (or exemplifiable) under these concepts; it is not done by imagining that they are attributing some universal properties (honesty, courageousness, tallness, redness, whiteness, circularity, rectangularity) to particulars.[23] Talk of

[22] Tomasz Bigaj's (2012) analysis of predicates such as 'courageous' has a clear realistic bias favoring the existence of properties (p. 21). He considers the subject-predicate statement "Socrates is courageous." Bigaj says that it is natural to interpret this sentence as being true when the individual named by its subject "Socrates" possesses (exemplifies) the property represented by the predicate "is courageous." More precisely, two semantical functions of linguistic expressions are usually distinguished: that of denotation and connotation. An expression denotes a thing if it can be truthfully said about this thing. The name "Socrates" denotes one individual, namely Socrates himself. On the other hand, the predicate "is courageous" denotes all individuals of whom it is true that they are indeed courageous. Thus, predicates typically denote more than one individual object. But each predicate also picks out a single object which is a universal. The predicate "is courageous" naturally singles out the property of being courageous. However, this predicate cannot be said to denote courage. Clearly it is not appropriate to say courage is courageous. Hence, we need to introduce the new semantical function of connotation. The predicate "is courageous" denotes courageous individuals, but connotes the property of being courageous. It is usually assumed that individual names such as "Socrates" do not connote anything. We don't use such names to pick out any property of an individual, but rather to pick out the individual as a whole. Bigaj says that postulating the existence of universals can help us analyze the basic structure of natural language.

[23] Critical comment: Bruce Aune (1985) states a contemporary 'One Over Many Argument' that postulates the existence of "determinate features" (properties, relations) present in relevant objects: "If several things fall under a common description, they must possess some absolutely determinate features by virtue of which the description is correctly applied to them. These ultimately determinate features, if they are not relations are in some sense "present in" the relevant objects, for it is by virtue of possessing them that the *objects* (and not something else) merit the appropriate descriptions" (p. 41). We reply, how do persons acquire *knowledge* of these 'determinate features'? There are different kinds of concepts and different kinds of entities (or objects) that these concepts can be about. Physical natural kind physical entities have determinate features (homeostatic) that require empirical investigation. Nontangible natural kind entities (e.g., knowledge, truth) have determinate features that require conceptual analysis. Group resemblance entities may have no necessary determinate features (e.g., game) and are determined by their subjective similarity(s) and are subject to conceptual analysis. Mathematical objects (as fixed definiens concepts) have their determinate features as a result of their (prescribed) axioms and definitions, and mutual consistency within a deductive system. Aesthetic knowledge ("x is beautiful") occurs when the determinate features of **x** truly engages **S**'s subjective mental affections. The view that 'knowledge' must be about 'objects' with 'determinate features' is flawed in its generality about objects and properties. The 'One Over Many Argument' arguing that if several things fall under a description, they must *possess* some *absolutely determinate features* by virtue of which the description is correctly applied to them, can be dismissed.

'properties' may be helpful and informative in some cases and domains (e.g. mathematics, physical science, in some ordinary talk),[24] but it is strongly cautioned that this language is often epistemologically misleading in other contexts (e.g. metaphysics, ethics, aesthetics, semantics, and other areas).[25] The conceptualist cautions that we should be aware (and beware) of language presented in terms of objects having properties.[26] Against Platonism, it seems intuitive that particular entities are *not composed of* (or are forms of) any independent universally existing abstract properties.[27]

V. The Conceptualist Worldview Explained

Like nominalism, conceptualism is a reaction against the platonic realist worldview.[28] The *Cambridge Dictionary of Philosophy* defines *'conceptualism'* as "the view that there are no universals and that the supposed classificatory function of universals is actually served by particular *concepts* in the mind." Fundamental to the

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[24] There are obvious references to properties in the physical sciences: brittleness, density, elasticity, electrical conductivity, fluidity, solubility, and so on.

[25] Critical comment: Does an *office* have the *property* of being *tidy* (or *not tidy*)? Soames (2010b, p.111, paraphrase) says that a way something (e.g., *my office*) could be, but isn't (e.g., *tidy*) is a *property*, it could have had, but doesn't. Attributing a *property* (e.g., untidiness) to an office is unneeded and extravagant. A better locution is 'I judge my office to be untidy (a group resemblance concept).' An office doesn't literally *possess* the *property* of being untidy. Compare this to judging that 'this office has *the property of possessing* one window' to the ordinary judgment that 'the office has one window.' The attribution of a *property possession* in the first sentence adds nothing to the sentence.

[26] Critical comment: In formal semantics, Tarski asserts that a sentence **p** (in context) has the *property* of being true or false (since it denotes or designates a state of affairs) and names a class of predicates. But do sentences have the *property* of being *true*? Is truth a property? Or is the *truth* of a sentence (in context) a *relation* (viz. correspondence) between '**p** asserted in context by **S'** and 'a state-of-affairs'? The use of the concept 'property' significantly affects a description of (linguistic) states-of-affairs.

[27] Critical comment: The idea that properties 'reside in' particular entities has generated theories that maintain that individual entities can possess 'essential' or 'accidental' properties. This essential-accidental distinction has generated bizarre claims that being a number is an essential property of 'eight'; while being the number of planets is an accidental property of 'eight.'

[28] Suppose that there is a physically instantiated pattern or structure which we haven't recognized. Does this imply that the mathematical pattern or structure is independent of us? No. Persons are needed to notice and posit formal definitions to describe the pattern/structure that is (loosely) identifiable as a phenomenon. The patterned nature of objects exists independent of us, but the mathematics used to measure it, requires the use of mathematical linguistic terms, within whatever axiomatic system best measures the pattern.

conceptualist worldview is that *everything that exists is spatiotemporal* (or derivative of it). Conceptualism views the existence of mathematical entities as dependent upon their definitions formulated as (physical) concepts within human brains. Mathematical entities (e.g., 3, rectangle, functions, sets) as so-called 'objects' are concepts in our head.[29]

'*Abstract entities*' are *redefinable* as *'mind-dependent creations or constructions.'* This definition is consistent with the ordinary language reportive definition of this concept and is contrary to the supernatural platonic definition that states that 'abstract objects' are 'entities that do not exist in space or time and are entirely non-physical and non-mental.' This proposed definition of 'abstract entity' is a pragmatic (or technical) stipulation of a preexisting concept. That is, it is a refinement, or explication, or engineering of an existing concept found in natural language and in the history of philosophy. The term 'abstraction' found in some dictionaries suggests that 'abstract entities' or 'abstract objects' are 'products of the mind.'[30]

For the conceptualist, it is intuitive that the Pythagorean theorem is a consequence of fixed definiens concepts and exists as a mental concept. This theorem does not have a single spatial location, but it's not non-spatiotemporal: it exists in any person who understands it (e.g., professional mathematician or laypersons). Likewise, the board game of chess has similarity to a mathematical object as it was invented, and its rules are grasped as a complex abstract object. The game of chess clearly isn't a single physical object, and so, it must be an (abstract) mind-dependent creation or construction, residing in players' minds. For non-players of chess, the concept of chess exists simply as 'a kind of board game,' without knowledge of the detailed rules of play.

Concepts can be said to 'exist in the heads of persons' but their (possible) instantiations/extensions may not exist. The 'concept' of **x** can be said to exist, if it is

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[29] It can be noted in passing (as a linguistic or psychological fact) that 'object' often connotes *unity* in an independent thing (with a definite structure and a clear-cut, sharp individuation). 'Entity' in contrast, oftentimes can connote a looser, more tentative identification of a thing-of-interest.

[30] On the definition suggested here, there are numerous kinds of 'abstract entities': all fixed definiens terms and concepts in mathematics, the precise definiens of group resemblance entities (e.g., chair, sports bar, crucial thinking, art) as guided by conceptual analysis; and the definiens of fictional entity terms. These are all mind-dependent constructions. Linguistic forms (e.g., definite descriptions, proper names, sentences) and the concept of a 'proposition' (no matter how defined), are likewise mind-dependent creations.

(consistently) instantiated in S's brain, even if **x** is non-instantiated in the actual world. For example, the concept of a 400-story hotel, as a definite description, may reside as mathematical possibility, and as a fuzzy mental image in anyone's brain, but this concept has no instantiation. At one time, the concept of 'humans flying in planes' was an un-instantiated concept. The concepts of 'a fountain of youth' and '*a priori* knowledge' can be articulated in some manner but may or may not have an instantiation. Newly minted fixed definiens concepts such as 'singular term,' 'propositional attitude,' and 'rigid designator' have been created at-will. *Concepts* (and their potential objects) when used in a context are *not always a representation of reality* (or a state of affairs).

The worldview intuitions of a conceptualist differ radically compared to Platonists and other realists, with respect to the possibility of the 'eternal existence' of mathematical entities and propositions. With formally defined entities and algorithmic systems, the conceptualist *denies* that such entities have 'always existed' independent of persons. If humanity should suddenly cease to exist, the proposition 2 + 2 = 4 would no longer be true-in-a-language, nor would it be true of sets of physical entities on earth. Physical entities (such as tigers, mountains, and so on) could remain existent, but no (true) mathematical propositions would remain. There would be no human cognizers to conceive of or empirically measure states-of-affairs such as the existing physical spatial union of two tigers encountering two different tigers, leading to there 'being four tigers.'[31]

When applied to the physiology of human beings, as stated above, physicalism is adopted in this essay as the view that mental states (e.g., *different kinds* of *concepts*) are *located* functionally as physical states. This view is an alternative to the philosophical claim that mental states are *identical* to physical states, and it opposes realist claims that there exist any supernatural entities, such as non-spatiotemporal abstract objects, Platonic forms, non-corporeal minds, immaterial souls, or gods. Different kinds of concepts such as fixed-definiens, group resemblance, natural kind, fictional, and proper names are tied to our mental manipulation of these kinds of concepts into a substantive thought. In

[31] Compare the opposing intuitions of Oystein Linnebo (2017): The necessity of the truths of pure mathematics are counterfactually independent of us humans, and all other intelligent life for that matter: "That is, had there been no intelligent life, these truths would have remained the same" (pp. 8-9).

mathematics, the conceptual knowledge of stipulated vocabulary, syntax, axioms, and inference rules of a deductive system leads to conclusions (given an input) that are true.

What Mathematical Entities Are About

In mathematics and physical science, there are legitimate normal discussions about objects, properties, relations, and identities. In a mathematical language, mathematical entities may be identified by their properties, and the properties of objects may be what compose their identities, and such entities stand in arithmetical relations. Although a conceptualist believes that the *existence* of all so-called '*mathematical objects'* are *based upon definable* (i.e., fixed definiens or implicit) *mathematical concepts;* this position doesn't imply any change to routine professional scientific and mathematical discourse (about objects, properties, relations, and identities). The philosophical 'deflation' of mathematical 'objects' to 'concepts' is irrelevant to the day-to-day thinking of mathematicians.

As an anti-realist position, a conceptualist may adopt a structuralist view of natural numbers, that respects the axioms of set theory: Numbers have no properties other than those they have in virtue of being positions in a x-sequence. 'Sets' are the principal objects of a formal mathematical-logical theory. The concept of 'identity' (and 'difference') makes sense in the context of a structure given by a theory. For instance, in arithmetic, there is nothing more than being the number 3 than having certain intra-structurally defined relational properties, such as succeeding 2, being half of 6, and being prime. The number 3 is identified as the successor of 2, the predecessor of 4, and so on, but not intrinsically. From set theory, the number 3 can be the measure of all 3-membered sets. 'Set' ('class') theory can be conceived as the singular concept implicitly defined within axioms, from which all other mathematical concepts could be defined.

Let us recite some intuitions. In natural language situations, with a 'speaker theory' of reference, the meaningfulness (i.e., intelligibility) of typical statements using *numerals* (e.g., 'I have *three* apples'), is not *about* a metaphysical 'object' as 'the number 3' (e.g., as a transcendent entity), but is interpreted as *about* the *items* (i.e., particular objects) being *counted*. When I say that 'I have three apples' my speaker *reference* of 'three' is the (set of) apples in my hands. A mathematical term, the numeral 3, when used in a context, is *not* the representation of a class of objects (i.e., where numbers are objects), nor is it wholly a singular term that refers to an object 3.

Further, it is clear that 'my mentally-physically instantiated concept of 3' isn't to be identified as 'the concept of 3.' Analogously, my 'mentally-physically instantiated concept of the Eifel Tower' isn't to be identified as the Eifel Tower. Instead, my concept of 3 comes from knowing its position, or its proper name, in a numerical sequence. My concept of 'the Eifel Tower' comes from seeing pictures and knowing some geography.

The sentence '3 is prime' is clearly not a report about my personal physical/mental fixed-definiens concepts of '3' and 'prime' but it is a truth within arithmetic. The linguistic terms in '3 is prime' and the sentence itself, are not about an independent reality. The sentence is true because of the fixed definiens concepts contained within it, and within a given mathematical language. The sentence '3 is prime' is neither objectively true, nor is it independent of a mathematical language. An account of the compositional mechanisms of the semantic reference of numerals to numbers (as objects) is only a problem for misguided semantic theorists.

Conceptualism Summarized

To summarize, the conceptualist responds to questions regarding 'aboutness' and the 'ontology' of mathematical theories by maintaining that there *exist* certain kinds of 'abstract' objects (e.g., mathematical entities, possible worlds) but these objects do *not* exist independently of us. Abstract entities are the product of human activity. With a theory of conceptualism, abstract objects are reconceived as mental *concepts* physically located within human brains. So-called 'abstract objects' should be redefined as 'abstract concepts' as being certain ideas-concepts-definitions in our head. Abstract '*objects'* are better understood as physically instantiated *'concepts.'*

The appropriate question is *not whether certain concepts exist*, but rather *does the use of a concept* show itself to be *fruitful* for accurately and informatively measuring a given domain. In practice, it is clear that the fixed definiens concepts of 'relative humidity,' 'gross domestic product,' 'validity,' and 'triangle' help explain and measure phenomena within their physical, social scientific, and mathematical domains.

A conceptualist theory clarifies the ontology of mathematical concepts as being a series of stipulative definitions and axioms (containing undefined terms) in a formal system. Logical pluralism is endorsed here as the view that there is more than one system of logic. Some formal representations may be more practical for certain purposes. We still use Euclidian geometry to measure 'parallel items' in ordinary everyday circumstances, even though non-Euclidean geometries are more informative for larger outer-space measurements. Besides practical measurement formal systems, there are pure systems that have no application. Any consistent axiomatization is at least worthy of mathematical study. Logical consequence is relative to a theory, model, or structure.

The Application and Usefulness of Mathematics

It has been asked how is mathematics applicable to questions found in the world? How can mathematical and artificial languages as a system of stipulatively-defined concepts measure various theoretical domains? The pragmatic answer endorsed here is summarized by Mary Tiles (2003) when discussing geometry and arithmetic:

> When we invent measures, discriminate between shapes and become, for practical purposes, interested in spatial relations, for example, or the relative motions of planets, mathematics can help us describe in systematic ways the various kinds of orders that we are interested in and to coordinate our various measuring units. Thus, astronomy uses geometry and arithmetic to help connect the various temporal units- such as day, lunar month, and solar year- in ways that make construction of a calendar possible. Geometry applied to measurements made on the ground makes building plans and maps possible. Mathematics is useful because measurement is useful, and this utility arises out of the practical contexts in which measures were devised (p. 351).

That mathematics is a tool (or instrument) for solving practical physical problems, is confirmed by Morris Kline in *Calculus: An Intuitive and Physical Approach* (1977). He notes different branches of mathematics and emphasizes the practical over the deductive:

> Elementary algebra was created to find answers to simple physical problems which in mathematical form called for solving first, second, and higher degree equations with one or two unknowns. Plane and solid geometry originated with the need to find perimeters, areas, and volumes of common figures and to state conditions under which two figures, say two triangles, are congruent or have the same shape—that is, are similar. Trigonometry, introduced by astronomers, enabled man to determine the sizes and distances of heavenly bodies (p. 1).

> The calculus was created in response to scientific needs, and we should study many of the applications to gain appreciation of what can be accomplished with the subject; these applications also give us insight into the mathematical ideas… Complete proofs of all the theorems are difficult to grasp when one is beginning the study of the subject… A thoroughly sound deductive approach to the calculus, one which the modern mathematician would regard as logically rigorous is meaningless before one understands the ideas and the purposes to which they are put. One should always try to understand new concepts and theorems in an intuitive manner before studying a formal and rigorous presentation of them. The logical version may dispose of any lingering doubts and may be aesthetically more satisfying to some minds, but this is not the road to understanding (pp. 5-6).

With the comments of Tiles and Kline in mind, it is best to understand mathematical systems as formal systems invented (or constructed) by mathematicians and physical scientists to help explain and precisely measure the nature of physical entities.

Let us consider physics. Physics has produced a classification of entities distinguished upon their intrinsic properties and external relations with other objects. Let us look at the theoretic definitions about two kinds of forces that are supposed to objectively exist; 'torque' and 'centripetal force.' We intuitively understand *'force'* as a push or pull. *'Torque'* can be equivalently defined as either of the following:

> "Torque" is the tendency of a force to rotate an object about an axis, fulcrum, or pivot.
>
> "Torque" is rotational force or the ability to overcome resistance to rotation. It is the cross product of force and radius. It is the amount of force applied tangentially to a circle.

Torque measures how hard something is rotated. For example, imagine a wrench trying to twist a nut or a bolt. The amount of 'twist' (torque) depends on how long the wrench is, and how hard you push on it. The amount of torque depends on (1) the force applied and (2) the length of the lever arm, and (3) the angle between the two. In symbols, a theoretic definition of 'torque' is:

$T = r \times F$
$T = r F \sin 0$

where T is the torque vector and T is the magnitude of the torque.
r is the displacement vector, and r is the length of the lever arm vector.
F is the force vector and F is the magnitude of the force.
x denotes the cross product.

0 is the angle between the force vector and the lever arm vector.[32]

This symbolic quantification of the magnitude of torque allows an objective solution to the following physics problem:

> In a hurry to catch a cab, you rush through a frictionless swinging door onto the sidewalk. The force you exerted on the door was 50N, applied perpendicular to the plane of the door. The door is 1.0m wide. Assuming that you pushed the door at its edge, what is the magnitude of the torque on the swinging door (taking the hinge as the pivot point)?

A physics student would solve this problem using the above symbolic definition of 'torque' and compute that T= (1.0m)(50N)sin(90). The magnitude of the torque is 50Nm. The definition of 'torque," as the physical rotational force, or the ability to overcome resistance to rotation, is quantified in the seven-line definition above. We can measure (and predict) exactly (measured in meters or feet) how far a door would swing open, given the 'force' (i.e. push or pull) that is exerted on the door.[33]

VI. Conclusion

With the doctrine of conceptualism and the method of conceptual analysis, we have denied that an understanding of the nature of mathematics is best approached from a metaphysical stance. The ancient idea that there is a metaphysical 'science of being' with a single sense of 'existence' is misguided. Instead, there are domains of physical existents, social scientific existents, mathematical existents, and fictional existents that are defined by *concepts* that are mutually, and individually, developed in our mental

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[32] It should be noticed that this definition of 'torque' is a contingent mathematical formula about a physical world phenomenon. It is a theoretic definition. It differs, for example, from a stipulative mathematical definition of 'derivative' as a formula developed as a fixed definiens concept. See chapter seven (vol. 1).

[33] Another example from physics of a related theoretic definition is 'centripetal force.' Centripetal force is a force that makes a body follow a curved path; it is always directed orthogonal to the velocity of the body, toward the instantaneous center of the curvature of the path. The magnitude of a centripetal force is defined F= (m)(v)squared divided by r; where F is the magnitude of centripetal force, m is mass, v is speed, and r is radius of curvature. The lessons here are: (1) that 'torque' and 'centripetal force' have definitions that describe the nature or associated material conditions, with respect to these forces. Torque and centripetal forces objectively exist and are theoretically definable and measurable. (2) the definitions in calculus (based on set theory axioms) are fixed-definiens concepts as units of measurement, that were historically intuitively devised to help answer (i.e., quantify) the answers to physical questions.

systems. As Aristotle and others have maintained, questions about identity and existence can only be informative with respect to specialized disciplines or in restricted contexts.

We have answered the question 'Do numbers exist?' with the following reply: *Numbers exist* in the domain of arithmetical concepts. The number '2' is identified as the successor of 1, the predecessor of 3, and so on, but not intrinsically. From set theory, the number '2' can be the measure (or extension) of all 2-membered sets.

Mathematicians can define the fixed-definiens existents of their theories, without believing them to be 'objective' or 'discovered as an abstraction from an independent modal reality.' The metaphysical extravagances of mathematical realism and mathematical nominalism in effect, have left us uninformed about the actual structures of axiomatized deductive systems. A philosophy of mathematics is informative when it reveals the structure of axiomatized systems (i.e., with vocabulary, grammar formation rules, inference rules, and semantics). In this way we can understand the nature of various mathematical languages.

'Game formalism' as a mathematical epistemology presented in chapter seven (vol. 1) combined with ontological conceptualism, allows us an elementary understanding of mathematical knowledge. On the game formalist account, mathematical knowledge is conceived as similar to knowing the rules of a game and making moves that accord within the rules of the game. If one adopts certain rules, then there are certain valid conclusions or outputs that follow, given certain inputs. This epistemic-semantic view of entailed mathematical truths is similar to what has been called 'syntactic formalism,' 'deductivism,' or 'game formalism.' Besides stating that the axioms of a deductive system express implicit definitions (and primitive terms) independent of any derivation from other propositions, formalism holds that deduced mathematical 'truths' are the consequence of following a consistent set of manipulation rules. Reasoning proceeds based upon syntactically marked regularities of expressions without an immediate concern for semantics. Propositions entailed from a proof are derived relative to a system's foundations (axioms, definitions, inference rules, grammar, and vocabulary). The definitional structure, ontology, and epistemology of mathematics is understandable.