Chapter Five

On Why Mathematics is Neither Analytic Nor *A Priori*

Abstract: This chapter critiques the concepts of 'analytic proposition' and *'a priori* knowledge.' It will be shown that the concept of 'analyticity' can plausibly be redefined to allow its classic core examples to remain as examples of analytic assertions (e.g., 'all bachelors are unmarried'), but with the redefinition, the concept of analyticity has limited extensions and no theoretical importance. With respect to *a priori truth*, it has been indicated in previous chapters that this is a very dubious concept. In this chapter, it is suggested that mathematics may be *knowable a priori*, but this knowledge is only about truths-in-a-language, and that with the descriptive-prescriptive distinction, the epistemic concept of the *a priori* can be alternatively explained and abandoned.

I. Introduction

Mathematics has been described as 'semantically analytic' and 'epistemically *a priori*.' Below, I argue that analytic truths are very limited in number (based upon a revised precise definition) and are theoretically insignificant. Following that, I argue that instances of so-called *a priori* truths are better explained using the descriptive-prescriptive distinction. Mathematics is neither analytic nor *a priori*.

II. The History of Analyticity

The analytic-synthetic distinction was initiated with David Hume's (1748) claim that all objects of human reasoning were either *'relations of ideas'* or *'matters of fact.'* On the face of it, this seems to be an intuitive distinction for explaining the difference between mathematics and physical science. Mathematics is intuitively a relationship of ideas and concepts, in contrast to the physical sciences, where scientists provide theories and hypotheses about what is empirically true.[1]

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[1] This chapter benefits greatly from *Analyticity* (2010) co-authored by Cory Juhl and Eric Loomis. Excerpts from their text and quotes from Kant are discussed here. In the first chapter Juhl & Loomis (J&L) provide a concise history of 'analyticity.' I highlight their observations and make interpretations and comments as needed.

Kant's Concept of Analyticity

Immanuel Kant (1781) responds to Hume. Kant believed that what distinguished the 'physical sciences' from the 'mathematical sciences,' was that physical science is about 'contingent' matters (e.g., knowable as *a posteriori* empirical truths) while mathematical science was concerned with 'necessary' truths (i.e., knowable as *a priori* truths). Kant thought that judgments like 7 + 5 = 12, with no experimental means for testing their truth, must be necessarily true and independent of sense experiences (i.e., *a priori*). For Kant, 7 + 5 = 12 is knowable *a priori*. Juhl and Loomis state that Kant's explanation leaves open some questions:

> J&L (p. 6): What is it that makes a statement necessary, and how is the truth of necessary judgments knowable *a priori*? This is where Kant introduced a distinction between two different types of judgments, analytic judgments and synthetic judgments. The explanation of what makes a statement necessary, as well as knowable *a priori* differs for the two types of judgment… Kant's most famous characterization of the distinction, the 'containment characterization,' occurs at the start of his *Critique of Pure Reason*:
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> Kant: *In all judgments in which the relation of a subject to the predicate is thought… this relation is possible in two different ways. Either the predicate B belongs to the subject A as something that is (covertly) contained in this concept A, or B lies entirely out of the concept A, although it does indeed stand in connection with it. In the first case, I call the judgment analytic, in the second synthetic. (B11).*
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> J&L (p. 6): Kant's own example of an analytic judgment is 'All bodies are extended.' Here Kant conceived of extension as the 'predicate' of the subject 'body.' In this judgement, Kant said, I do not require to go beyond the concept which I connect with "body" in order to find extension bound up with it. By contrast, Kant thought that when I say, "All bodies are heavy," the predicate is something quite distinct from anything I think in the mere concept of body in general, and the addition of such a predicate therefore yields a synthetic judgment.

Juhl and Loomis note that Kant's 'containment explanation' of 'analytic judgments' opens more questions: "In what sense is the concept extension contained in the concept of body? Clearly, it is not the same sense as which the predicate 'black' is explicitly contained in the subject 'black horse' in the judgment 'A black horse is black.' The containment in Kant's example is 'covert.' But what does 'covert' mean here?" (p. 7).

Kant's account of analyticity was intended to provide an explanation for the necessary truth or necessary falsehood of some statements. In the case of necessary truths or falsehoods that are analytic, their necessity derives from the simple fact that the predicate is contained in the subject. To deny this is to violate the law of non-contradiction (i.e., **p** v not-**p**), a seeming necessary truth. Analytic claims are those whose denial leads to self-contradiction. But Kant didn't think that every necessary truth is analytic, including the judgment of 7 + 5 = 12. This judgment is a 'synthetic necessary truth' as explained by Kant with the following:

> Kant: *We might, indeed, at first suppose that the proposition 7 + 5 = 12 is a merely analytic proposition that follows by the principle of contradiction from the sum of 7 and 5. But if we look more closely we find that the concept of the sum of 7 and 5 contains nothing save the union of the two numbers into one, and in this no thought is being taken as to what a single number may be in which combines both… Arithmetical propositions are therefore always synthetic. This is more evident if we take larger numbers. For it is then obvious that, however we might turn and twist our concepts, we could never, by mere analysis of them, and without the aid of intuition, discover what is the sum (B15-16) (p. 8).*

> J&L (pp. 9-10): The notion of intuition that appears in this last remark holds the key to Kant's account of the necessary but synthetic propositions of mathematics and geometry. Such propositions are necessarily true, and known *a priori* in virtue of what Kant called a 'construction in intuition' (A720/B748). Kant's idea, in the case of mathematics and geometry, was that we have certain 'pure' *a priori* intuitions of space and time. 'Intuitions' in Kant's sense were a kind of immediate relation of the mind with an object of knowledge. This relation is 'immediate' in the sense that it is a direct relation with the object of knowledge unmediated by signs, marks, or concepts (cf. A19/B33, A25/B40). The intuitions of space and time are not sensory, Kant believed, but instead constitute part of the basis for our sensory experience. His idea here was that space and time were not *discovered in* experience but rather *presupposed by* experience… Our intuitions of space and time thus were *a priori* conditions of possible sensory experience, Kant thought, and furthermore the conditions that made possible the synthetic *a priori* truths of geometry and mathematics… Mathematical propositions, like geometric ones, derive their status as *necessary* and *a priori* from the role that the *a priori* intuition of *time* plays in making our experience possible (A719-20/B747-8).

Juhl and Loomis state that "the complexity of Kant's work concealed ambiguity and vagueness at certain crucial points, and left many open questions" (p. 11). The lack of clarity in Kant's notion of 'analytic truth'[2] also raised questions among many academics, including Bernard Bolzano (1837). With Bolzano's analysis and correction of Kant's 'figurative modes of speech,' he suggested that analytic statements ought to include those that are logically true, such as his examples of the laws of identity (A is A), or the principle of the excluded middle (every object is either B or not-B).

Frege's Concept of Analyticity

Gottlob Frege likewise believed Kant's conception of analyticity was too vague. Frege hoped to show that arithmetical propositions were not synthetic *a priori*, but analytic. Frege regarded analytic truths as those which have a proof resting solely upon general logical laws and definitions. 'Analytic truths' are just those statements which are *general laws of logic (i.e., axioms)* or *are derivable from those laws alone (i.e., theorems)*. Frege's motivating idea was to show how 'definitions of number' and the 'basic laws of logic' could be combined so that all mathematical truths were analytic. Mathematics could be reduced to logic (i.e., 'logicism'). Logicism rejected the relevance of Kant's synthetic *a priori* intuitions about time and maintained that arithmetic contains only analytic truths.

While Frege's mathematical logicism failed under Russell's paradox, several members of the Vienna Circle picked up on Frege's notion of analyticity, especially Moritz Schlick and Rudolf Carnap, who proposed that the ‘truths of logic’ expressed *conventions* governing a given language.

The Vienna Circle's Concept of Analyticity

The Vienna Circle was a group of scientists, mathematicians, and philosophers who met in Vienna from 1928 to 1936. Moritz Schlick (1882-1936) was among the forefront of philosophers who conceived of analytic truths as an expression of the conventions governing language. Schlick (1925) adopted discoveries made in axiomatic

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[2] As Juhl and Loomis state, Kant's *'containment'* definition is different from a *'contradiction'* concept of analyticity, that he also endorses, whereby 'I merely draw out the predicate in accordance with the principle of contradiction, and thereby at the same time become conscious of the necessity of the judgment.' (A7). Kant offers two different characterizations of the ‘analytic.’

theory made by mathematician David Hilbert. Hilbert had shown how the terms 'point' and 'line' were implicitly defined within axiomatic Euclidean geometry. Schlick saw in Hilbert's methods the possibility of implicitly defining certain concepts within the context of a formal (axiomatic) system. Juhl and Loomis quote Schlick and then comment:

> Schlick: *A system of truths created with the aid of implicit definitions does not at any point rest on the ground of reality. On the contrary, it floats freely, so to speak, and like the solar system bears within itself the guarantee of its own stability. (p. 37).*
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> J&L (p. 21): A system of truths is in part 'created' through stipulation, Schlick thought. Here Schlick was referring not to empirical or synthetic truths, but to formal, *a priori* ones. We could guarantee the stability of such a system if we could find a consistency proof for the axioms (ibid., 357). Such a stipulated, provably consistent system would 'float freely.' There would be no need for an 'explicit' definition of the primitive terms that connected them with the empirical world. *A priori* knowledge was thus conceived by Schlick to consist of stipulations of symbol systems that implicitly defined a set of concepts.

Juhl and Loomis state that these ideas allowed the Vienna Circle to account for several characteristics of analytic propositions (p. 21):

> 1) Analytic truths were tautologous principles of language use, not descriptions or reports of observations.
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> 2) The *apriority* of analytic truths was accounted for by their being the conventions of language which as stipulations were known as *a priori*, or at least knowable without appeal to experiences other than those required for understanding the language.
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> 3) The seeming *necessity* of analytic truths such as the laws of logic was accommodated by their being akin to rules or axioms for the use of a particular language.
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> 4) The *scope* of analytic truth could be greatly expanded by treating them as conventional stipulations. For not only would the laws of logic and mathematics be analytic, some propositions of science, such as those defining law-governed expressions such as 'force' or 'simultaneity,' would be as well.

Classes of synthetic *a priori* propositions as conceived by Kant, were now explainable as analytic truths, with the exception of metaphysical propositions which were thought to be unverifiable and meaningless. Juhl and Loomis sum up the view of Schlick and Carnap:

> J&L (p. 21): Analytic truths could be treated as 'tautologies,' statements which do not say anything about the world, but which instead express logical properties among concepts or among statements. These tautologies were understood to be conventional stipulations that governed the use of a given language by telling us what words mean, or what statements could be inferred from others. Knowledge of the truths of mathematics, geometry, metaphysics, and other supposedly synthetic *a priori* truths were instead seen as expressions of the analytic 'tautologies' that governed our use of language. This allowed Vienna Circle members to explain our knowledge of such truths without appealing to intuitions or to otherwise 'mysterious' faculties of knowing. The true statements of mathematics and geometry are indeed necessary, and they are knowable *a priori*, but only because they express conventions of language. They say nothing about the world itself; to know facts about the world, we must turn to experience. Thus, the Vienna Circle supplied a theory of analytic truth which they believed to be compatible with empiricism.

The Vienna Circle used analyticity to give an account of *a priori* knowledge where the truths of logic and mathematics were thought free of dubious metaphysical commitments.

Carnap's Conceptions of Analyticity

Rudolf Carnap's book, *The Logical Structure of the World* (1928, the '*Aufbau*') was concerned with the epistemology of the physical sciences. Carnap shared Bertrand Russell's interest in philosophy as a scientifically informed investigation of the logical forms of the statements of science. Carnap's definition and 'adequacy conditions' of analyticity underwent several significant changes. The unity of his formulations, however, was that analytic truth is a *language-relative* notion. In the *Aufbau,* like Frege and Russell, Carnap believed that there was a *single logic* that underlies all reasoning, including physical science, mathematics, or philosophy

In *The Logical Syntax of Language* (1937), Carnap made a major revision. He no longer assumed a single, universal logic, and adopted a 'Principle of Tolerance':

> *In logic, there are no morals. Everyone is at liberty to build up his own logic, i.e., his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments. (1937, p. 52).*

Carnap thought that in the formulation of a language, there should be a specification of 'use' of signs. Carnap wanted to build formal languages of precisely defined rules that formed sentences. He says:

> *The aim of logical syntax is to provide a system of concepts, a language, by the help of which the results of logical analysis will be exactly formulable. Philosophy is to be replaced by the logic of science—that is to say, by the logical analysis of the concepts and sentences of the sciences. (1937, xiii).*

Carnap's conception of analyticity later changed with consideration of Alfred Tarski's conceptions of 'object language' and 'metalanguage.' He engaged in conversations with both Godel and Tarski. Carnap accepted that everything mathematical can be formalized, but mathematics cannot be exhausted by one system; it requires an infinite series of ever richer languages. Truth can only be defined in a metalanguage having more expressive resources than the object language.

In the *Introduction to Semantics and Formalization of Logic* (1942) Carnap's use of Tarski's methods led him to a new problem of defining 'L-truth.' He continued to distinguish 'analytic statements' which formulate rules for a language or expressed consequences of those rules, and 'synthetic statements' that were based upon extra-linguistic matters of fact. He proposed several methods of explicating analyticity including reference to the notion of a 'state description' in his book, *Meaning and Necessity, a Study in Semantics and Modal Logic* (1956):

> *a 'state description' gives a complete description of a possible state of the universe of individuals with respect to all properties and relations expressed by the predicates of the system. Thus, the state-descriptions represent Leibniz' possible worlds or Wittgenstein's possible states of affairs. (1956, p. 9).*

State-descriptions are thus conceptualized as complete descriptions of possibilities; of possible ways the world could be. Carnap then proposed the following 'adequacy condition' in order to give a definition of L-truth:

> *Convention. A sentence Si is L-true in a semantical system S if and only if Si is true in such a way that its truth can be established on the basis of the semantical rules of the system S alone, without reference to (extra-linguistic) facts. (p. 10).*

This 'convention' gives us a condition that any definition of L-truth must meet but it is not a definition of L-truth. But Carnap thought that *state-descriptions* could allow us to *define L-truth*, and *thus analyticity*, for *given language S1*:

> *Definition. A sentence Si is L-true (in S1) = (df) Si holds in every state description of (in S1). (p. 10).*

With the resources of a metalanguage to characterize the properties of an object language, this allowed Carnap to give a definition of 'logical truth' (or L-truth) which he treated as synonymous with 'analytic.' Carnap captured the Vienna Circle's idea that an 'analytic sentence' is true, no matter what the empirical facts may be. He included mathematical truths among analytic truths. Juhl and Loomis summarize Carnap's thinking:

> J&L (p. 75): Carnap introduced his 'Principle of Tolerance' according to which there are no 'morals' in logic. Rather, we can freely construct logical systems and 'languages' consisting of inference rules, axiom systems, and whatever else we please without concern for whether such languages are 'true' or 'correct.' Carnap thought that traditional philosophy could be replaced by the construction and study of such languages, with our goal being that of resolving traditional philosophical disputes by building and agreeing upon formal language systems that embody our preferences through our choice of linguistic rules. These rules and their consequences, Carnap thought, are the analytic truths of the language. They are statements which are true solely in virtue of the linguistic system itself. Carnap thus replaced his earlier efforts to 'reduce' scientific knowledge to subjective experience with the construction of formally precise languages…(and) responded to Godel's incompleteness results for a given language by allowing the use of a richer meta-language.[3]

[3] Benjamin Marschall (2022) says that it is crucial to appreciate what Carnap means by 'languages.' Marschall (p. 2): In most contexts Carnap refers to what he would later call '*linguistic frameworks'*: formalized languages with explicitly stated rules of use for the expressions provided. In such a language, some sentences are derivable from the rules themselves, just as logical rules can be derived in a calculus without any additional premises. These sentences are classified as *analytic*, whereas sentences that are independent of the rules are *synthetic*. Empirical claims about the observable world are the paradigmatic examples of synthetic statements (Carnap, 1937, p. 41). But analytic statements are meaningful too, despite their being inferentially isolated from statements about observations (1937, pp. 318f). And Carnap sets up the languages he constructs in such a way that mathematics comes out as analytic. Mathematical questions are answered through "logical analysis based on the rules for new expressions" (1956, p. 209).

There are difficulties with Carnap's thought.[4] Without going into further detail about Carnap's programs concerning the logic of the physical sciences, the structure of language, and eliminating metaphysics[5], it is concluded that Carnap's quest for an explication of 'analyticity' wasn't successful. What is important is the recognition of an intuition that shaped Carnap's and the Vienna Circle's concept of analytic propositions: Analytic sentences are true-in-language and are (thus) knowable *a priori*.

It is also important to understand that Carnap sought an 'explication' (or a 3c 'precise formal definition') that supports and explains the idea that 'analytic truths' or 'necessary deductive truths' are different from 'contingent empirical truths.' An 'explication' was understood by Carnap as the process of replacing an inexact or vague concept by an exact and precise one, ideally within the context of an artificial, precise language. With a theory of 'analytic truth' (or 'L-truth') Carnap sought an account of this concept that is explanatorily consistent with other epistemic and semantic concepts such as knowledge, empirical, necessity, truth, synonymy, reference, and so on. Carnap's final post-theoretical world-view intuition was that *mathematics is composed of truths* that are dependent upon the structure of an adopted formal language (and thus, *analytic*), while in the *physical sciences we seek truths* that are independent and empirical (and *synthetic*). We have knowledge of analytic and synthetic truths and pragmatically devise systems of formal deductive language syntax to appropriately measure a physical domain.[6]

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[4] J&L (pp. 59-60) comment on Carnap's conception of state-description, convention, and definition: "There is an important limitation to observe here. A state-description, Carnap thought, contains every atomic sentence of a language and its negation (1956, p. 9). So, it is a 'complete' description of a possible state of the universe in a given language. This completeness requires any two atomic sentences be logically independent of each other, in the sense that no class of atomic sentences can logically entail the truth or falsity of another atomic sentence…The requirement of the logical independence of atomic sentences must be invoked to prevent contradiction. But it imposes a significant constraint on what languages that use the definition to explicate 'analytic' must be like. For example, such seemingly simple sentences of English as 'Point *p* is red at time t' could not count as atomic sentences. For this sentence entails the falsity of numerous other sentences like 'Point *p* is blue at time t'… Indeed, it is hard to know what descriptive predicates could appear in atomic sentences while meeting Carnap's independence requirement."

[5] Carnap was concerned to use the tools of modern mathematical logic and metamathematics to dissolve all 'metaphysical disputes' and replace them with a more rigorous, fruitful, and constructive project of language planning by developing and investigating varieties of formal deductive structures for application in the empirical sciences.

[6] A good exposition of Carnap's work is by Steve Awodey, "Carnap's Quest for Analyticity: The *Studies in Semantics*" among other essays in *The Cambridge Companion to Carnap* (2007) Friedman & Creath, eds.

A.J. Ayer's Classical View of Analyticity

After detailing Quine's famous objections to the notion of analyticity from a naturalist's perspective,[7] Juhl and Loomis present the central claims about analyticity that can be attributed to A.J. Ayer in his *Language, Truth, and Logic* (1946). Ayer presents a logical empiricist's account of analyticity with ideas that are consistent with the Vienna Circle (which Ayer visited as a young man), and elements found in Kant, Hume, Wittgenstein, and Frege. A list of eleven basic claims supported by Ayer, are stated by Juhl and Loomis (pp. 175-179) as follows:

> (1) Factual statements are all and only statements that can be justified by appeal to empirical evidence. A proposition is synthetic when its truth is determined by facts of experience.
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> (2) Some statements express necessary truths. They include equations and theorems of mathematics, statements of geometry, laws of logic, and certain generalizations such as 'no point is both red and green all over' which are not reducible to logic.
>
> (3) No statement can be necessarily true on the basis of empirical observation.
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> (4) No necessary truth would ever, under any conditions, come to be regarded as disconfirmed by empirical evidence.
>
> (5) Necessary or universally valid statements can be interesting or surprising.
>
> (6) A proposition is analytic when its validity depends solely on the definitions of the symbols it contains.
>
> (7) All and only analytic statements express necessary truths.
>
> (8) The truth of analytic statements is 'trivial' or 'obvious,' or can be determined from the statement alone.
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> (9) Denials of analytic statements lead to self-contradiction; and are therefore 'self-stultifying.'

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[7] In his "Two Dogmas of Empiricism" (1953a) Quine convincingly argued against the first dogma that there is a clear and sharp distinction between formal or analytic truth in logic and mathematics, as contrasted to a factual or synthetic truth in empirical natural sciences, and a second dogma that there can be a phenomenological reduction, which are at root identical. Since the analytic-synthetic distinction was central to Carnap's philosophy, Quine believed that Carnap's philosophy was plainly flawed.

> (10) Analytic statements are true by convention.
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> (11) Analytic truths are senseless, but in a way distinct from the way that metaphysical utterances are senseless. They elucidate or illustrate the way in which we use certain symbols, by indicating or showing the conventions or rules of syntax justifying those uses.

We should just take notice of Ayer's intuitions about 'facts,' 'necessary truths,' 'definition,' 'analytic propositions,' 'synthetic propositions,' 'validity,' 'self-contradiction,' and 'convention.' These are background beliefs in the search for a definition of analyticity.

III. Searching for a Precise Stipulative Definition of 'Analytic' Propositions

We now examine two standard definitions of 'analytic assertion' (or 'analytic sentence,' 'analytic truth') that will serve as the starting point for a study of analyticity.

> An **'analytic assertion'** is a sentence that is true solely in virtue of the meaning or definition of its terms.
>
> An **'analytic assertion'** is a sentence that is true (or false) solely by the conventions of language.

From the definition of 'analytic truths' as envisioned by the logical positivists in the Vienna Circle, it was conceived that the syntax, axioms, definitions, and inferences rules that are introduced in the formulation of a deductive system are *linguistic conventions* and that such *'analytic statements' are true by convention* (e.g., Ayer's #10). An analytic sentence is understood true (or false) in virtue of the meaning or definition of its terms. Logical propositions were explained as 'tautologies' and 'analytic.' Logic and mathematics are nothing but tautologies. The self-evidence of an analytic sentence was from its purely formal structure. These formal truths had no referential content. Analytic truths were tautologous principles of language use, not descriptions or empirical reports of observations. For the logical positivists, the concept of a proposition being 'analytically true,' 'tautologous,' '*a priori*,' or 'necessary' became nearly interchangeable.

What Are Analytic Truths? Are These Truths Theoretically Significant?

In chapter seven (vol. 1), it was argued that axiomatic structures consist of the following: (1) the introduction of a *vocabulary of symbols* that are the definitions of what counts as an individual constant, individual variable, predicate, proper name, sentential connective, punctuation, and quantifier, (2) the introduction of *syntactical formation rules* (or grammar) that defines how well-formed formulas are to be constructed out of symbols as a procedure for determining whether a sentence (i.e., finite strings of symbols) is well-formed and 'meaningful' or not, (3) a set of truth-preserving *inference rules* (e.g., *modus ponens*) and (4) a *semantics* (e.g., 'b' means Bill, and predicate 'S' means 'sings'). Behind this structure, are implicit *axioms* that underlie formal systems.

With the above analysis of the structure of deductive systems as a guide, I *propose the following alternative definition* (as a 3c mode of stipulative definition):

> An **'analytic assertion'** is a sentence that is true relative to and entailed solely by the fixed definiens definitions in the vocabulary and the syntactical rules of a (natural or artificial) language *without* the explicit use of any discrete *inference rules*. In other words, analytic sentences are true (in-a-language) just in virtue of their fixed definiens terms and syntax.

On this proposed fixed-definiens definition, core example propositions such as 'a triangle has three sides,' 'all squares are rectangles,' 'the square root of 3 is irrational' '3 is prime,' '3 is less than 4,' are *conceptually implied* from the *fixed definiens definitions* of 'triangle,' 'three,' 'sides'/'lines,' 'square,' 'rectangle,' 'square root,' 'rational,' 'prime,' 'less than,' and 'four.' In natural language, assertions such as 'all bachelors are unmarried,' 'the ninth chapter of a ten-chapter book is its penultimate chapter,' 'if today is Friday, then tomorrow is Saturday,' '**S** was born on her birthday,' and 'a whole is at least as large as any of its parts,' are all examples of analytic sentences. By the mere knowledge of the definiens of these concepts ('bachelor,' 'penultimate,' 'Friday', 'birthday,' 'whole,' 'part') we can *know* these truths immediately without the synthesis of any inference rules.

The *axioms* of mathematical deductive systems may be called 'analytic' under this definition. Axioms are 'true-in-a-language' as the result of the vocabulary and the syntax of a deductive system. As explained in chapter seven (vol. 1), Hilbert (1934) understood that syntactically an 'axiom' is a proposition composed of undefined primitive terms and it cannot be proved from other propositions (and axioms) within a formal system.

From an epistemic point of view, it was argued that an 'axiom' is a foundational prescriptive proposition that underlies a set of stipulative definitions, grammar-syntax, and inference rules that measure a specified domain. Axioms, such as those in geometry and arithmetic (e.g., Euclid and Peano) are only distinguished *after* the formal system (of definitions, grammar, and inference rules) has reached some high degree of sophistication (and utility). Axioms are generated from the vocabulary and syntax, and we may say they are prescriptively adopted as 'true-in-a-language'.

With this new definition of 'analyticity,' examples of analytic truths are *limited* in number. The definiens of the proposed definition of 'analytic sentence' in effect *marginalizes the importance* of 'analyticity.' This definition captures core examples of what have been called analytic sentences (e.g., 'bachelors are unmarried,' axioms) but without including all of the other kinds of mathematical propositions as being analytic truths. Although it has been said that the foundation of mathematics is based upon 'linguistic conventions' it is more precise to characterize it as a deductive science built upon certain kinds of structured *prescriptive* propositions, which may be termed as 'truths-in-a-language.'

In opposition to Quine (1936) and Ayer (#10 above), it is false that prescribed 'stipulated definitions' can be 'true' by convention. Stipulated definitions are not true by convention. It is fallacious that a 'stipulated definition' in any context should be conceived of as true (or false) because of a consensus of *human agreement*. A proposition is 'true' just in case it corresponds to states of affairs in the world *independent* of human agreement, and 'false' otherwise. Formal axiomatic systems should *not* be thought of as being composed of some set of true-by-convention propositions. Instead, they are composed of a set of systematic stipulative fixed-definiens definitions. The new definition of analyticity recognizes some sentences are 'analytic' without labeling all the other elements of a deductive system as being 'analytic.' Theorems, logical truths, and premised deductive entailments (e.g., 7 + 5 = 12) are not analytic under this definition.

(A) Theorems Are Not Analytic

The concept of analyticity, as narrowly defined above, doesn't include theorems as being analytically true. Theorems are the basic formulas of logical equivalence that are necessarily true-in-a-language relative to and entailed by the axioms, syntax, and inference rules of a formal language. The truth of theorems follows necessarily from axioms that are discharged in a proof. For example, the theorem that 'the sum of the interior angles of a triangle is 180 degrees' is a deductive consequence from the axioms of Euclidean geometry. In formal logic, the two equivalences stated in DeMorgan's Rule as a theorem follow in the same way from the axioms of a formal system. Because axioms are explicitly used as premises (or assumptions) and truth-functional inference rules are used in proving any theorem, their entailed truth doesn't follow exclusively from the fixed definiens of the concepts involved. Theorems are true-in-a-language, but they are not analytic.

(B) Tautologies (Logical Truths) Are Not Analytic

Tautologies (i.e., logical truths) are not analytic.[8] This is because tautologies are the immediate consequence (or entailment) from an axiom in conjunction with syntactic formation rules and truth-functional connectives (is, or, and, not, if… then). A number of prominent axioms are responsible for entailing tautologous truths:

1) Identity Axiom: For any object x, it is necessarily the case that **x** is identical with **x**.

2) Non-Contradiction Axiom: It cannot be the case that **p** and not-**p**.

3) Bivalence Axiom: Every declarative sentence/statement expressing a proposition (in a domain) has exactly one truth value, either true or false.

4) Excluded Middle Axiom: For every statement **p**, either **p** is true, or **p** is false.

5) Transitivity of Identity Axiom: Whenever a=b, and b=c, then a=c.

[8] Truth-functional tautologies are necessarily true in virtue of their syntactical sentential form where standard truth-functional connectives are used. A tautology is a proposition form that is true, regardless of the substitution instances inserted in any of its propositional variables. In propositional logic, a well-formed formula (wff) is a 'tautology' if and only if 'the wff is true for all possible truth-value assignments in its truth table.' Self-contradictions are just negations of tautologies. Tautologies and self-contradictions are recognizable by a purely formal mechanical process; that of truth-tables. A tautology is compatible with every state of affairs, thus it contains no information.

Tautologous truths such as 'Abraham Lincoln is Abraham Lincoln' assumes identity, 'it is raining or not raining' assumes excluded middle, and 'you don't know what you don't know' assumes non-contradiction. The sentences 'if **S** failed his statistics exam, then **S** hasn't passed his statistics exam,' and 'if **S** stopped singing, then **S** did not continue singing' are tautologies based upon their conditional form and the not-operator in the consequent. That 'if **x** is taller than **y**, and **z** is taller than **x**, then **z** is taller than **y**' is non-analytic because it incorporates the axiom of the transitivity of identity.

Tautologies are *not* just true in virtue of fixed-definiens definitions and syntax, but *in addition*, they are entailed necessarily true by relevant axioms.

(C) Premised Deductive Entailments Are Not Analytic

Premised deductive entailments are statements where a conclusion is *necessarily* true relative to and entailed by the 1) axioms, 2) vocabulary-syntax, 3) grammar-syntax, 4) inference rules and 5) the premises of a valid argument. On this new definition of analyticity, the mathematical proposition '7 +5 = 12' *is not an analytic assertion* because it is an entailed deductive truth from an inference rule (the addition function), with the *numerical quantities as premises* (or variables) in the equation. A mathematical proposition, such as this, is a *synthesis* of prescriptive rules and substantive premises. When 7 hypothetical objects (one premise) are added to 5 more hypothetical objects (second premise) these premises entail the conclusion that there are now 12 hypothetical objects. Similarly, in geometry, when a rectangle is 5 feet wide (a premise) and 4 feet narrow (a premise), those premises in conjunction with the axioms and definitions of Euclidean geometry entail a rectangle that is 20 square feet. The entailed conclusions of these statements (i.e., 12 objects and 20 square feet) require premises and inference rules and are not solely entailed by the fixed definiens of a language. Kant was right that sentences like 141678 + 639465 = 791143 should not be deemed analytic. This proposition is a synthesis of 'variables,' an 'addition function,' and an 'equality relation.'

Similarly, with a valid deductive argument, the structure of the argument makes it so that, *if* the premises are true, then the conclusion must be true. The following is an example of a simple valid argument:

Premise: If **p**, then **q**.
Premise: **p**.
Conclusion: **q**.

This argument is valid because *if* the two premises are assumed true, then the conclusion cannot be false. The necessity of **q** (given that both premises are true) *isn't analytic*, as defined above, because there is an explicit use of inference rules (viz. *modus ponens*).

Similarly, when a formally valid deductive argument is transformed into a single sentence of a conditional form (i.e., an argument's premises are combined into an antecedent and the conclusion is transformed into a consequent), the resulting material conditional is 'true-in-an-artificial-language' but cannot be termed analytic, because the use of axioms and inference rules are essential for its truth.

(D) Sentences Implied from Presupposition Are Not Analytic

Analytic sentences should not be confused with 'presuppositions' that are knowable based upon shared background empirical information. For example, the assertion 'all cats are animals' is not analytic. Given that **x** is a cat, it can be inferred that **x** is an animal. But 'cat' is a natural kind concept (and not fixed definiens). The concept of 'cat' doesn't entail that cats are animals. It is just a contingent empirical true proposition that cats are animals (and not sophisticated alien computers). Likewise, the sentence 'red is a color' is non-analytic because 'red' can be interpreted as either a natural-kind or group resemblance concept. 'Red' is not a fixed definiens concept because it doesn't unequivocally identify the items that may fall under its definition. Items that *appear* to be 'red' fall under a group resemblance according to a viewer's perspective, or 'red' may alternatively be identified as natural kind as stipulated by a range of wave lengths. The sentence 'red is a color' is true based upon the presupposed background knowledge of the speaker's use and definition of 'red' but it isn't analytic. Truths that 'cats are animals' and 'red is a color' are inferred (or presupposed) from reportive definitions and *not entailed* from fixed definiens concepts.

Analytic Sentences Are Theoretically Insignificant

Of course, theorists are free to propose another fixed definiens concept of 'analyticity' that allows theorems, tautologies, valid entailments, and presuppositions to be examples of 'analytic sentences' but such a definition is likely a disjunction of criteria to cover disparate sentences deemed as 'obviously' true. Analytic sentences are a small part of the nature of deductive necessity. The more important part of deductive necessity is that from a well-formed language of prescriptions, we can calculate derivations that can be described as 'true-in-a-language' (or a ‘logical consequence’ or 'entailed').

IV. Possible World Semantics, Metaphysical Necessity, and *A Priority*

Let us summarize the prescriptive propositions that make up the construction of a possible world semantics (loosely following Edwin Mares, 2011, pp. 14-22):

1) There is a set of possible worlds.

2) A 'possible world' is a way the universe might have been.

3) In different possible worlds, different propositions are true.

4) We say that a world W2 is 'nomically accessible' from another world W1, if and only if the laws of nature of W1 are true in W2.

5) In W, "it is physically necessary that S" is true if and only if S is true in every world nomically accessible from W.

*6) In W, "it is *metaphysically necessary* that S is true" if and only if S is true in every possible world.

7) In W, "it is possible that S" is true if and only if in some world S is true.

8) In W, "S1 and S2" is true if and only if S1 is true in W and S2 is true in W.

9) A 'proposition' is a sentence that is made in a context, so that the sentence and a context are what allow the evaluation of whether a sentence is true or false in a possible world.

10) A proposition is 'contingently true' if it is possible, but not necessary. A proposition is contingent, if it is true in some possible worlds and false in others.

11) A 'necessary falsehood' is true in no possible world.

Mares states that this 'possible worlds semantics' is the leading theory of 'necessity.' We can readily identify sentences 1 and 3 as prescriptive, while the other sentences are all understandable as stipulative definitions that form fixed definiens concepts. From Mares' perspective, however, these propositions about necessity are truths that are the result of *a priori* reasoning and are not subject to empirical observation.[9]

It is from the above possible world semantics, that the analytic sentence 'every bachelor is unmarried' becomes a paradigm example of something that is metaphysically necessary and can be known *a priori*, according to Mares (p. 5). Mares provides the following examples of five propositions that are supposed to be metaphysically necessary (p. 15): 'Red is a color,' 'Red is not a shape,' 'Every colored thing is extended,' 'Two is a larger number than one,' and 'If it exists, then planet Venus is identical to itself.'

Are There Cogent Examples of *A Priori* Truth?

A priori propositions are thought to be *known true* without any justification from the character of a person's empirical experiences. The truth of an *a priori* proposition is believed to be from 'reason' or 'pure thought' alone. As already emphasized in chapter 3, the existence of *a priori* truths is very doubtful, and the several examples of alleged *a priori* truths (i.e., Kripke's) were given new interpretations that didn't require an appeal to *a priori* truth. Below, I challenge Mare's examples of sentences that are allegedly knowable *a priori.*

[9] Many proponents of possible world semantics understand philosophical claims to *not* be just about what *is* the case (i.e., about how things are), but that philosophy seeks to assess how things *mus*t be. Philosophy is presumed to be study of necessary truths and what could *possibly* be true, and thus, the study of modality. Proponents of *a priori* knowledge include Alvin Plantinga (in *Warrant and Proper Function*, 1993, chapter six), George Bealer ("The A Priori" in *The Blackwell Guide to Epistemology*, Greco & Sosa (eds) 1999), Albert Casullo ('A Priori Knowledge' in *The Oxford Handbook of Epistemology*, Paul Moser (ed) 2002) and Laurence Bonjour ("In Defense of the a Priori" in *Contemporary Debates in Epistemology*, Steup, Turri, and Sosa (eds) 2014). Other essays are found in *New Essays on the A Priori*, edited by Boghossian and Peacocke (2000). Harty Field discusses "Recent Debates about the A Priori" in *Oxford Studies in Epistemology, Volume 1* (2005). I follow Edwin Mares' *A Priori* (2011) and Laurence Bonjour's *In Defense of Pure Reason* (1998) as comprehensive and clearly written monographs of the topic.

A Priori Knowledge of Analytic Sentences

The first class of alleged *a priori* propositions, according to Mares, are analytic sentences. Mares' examples of 'every bachelor is unmarried' and 'two is a larger number than one,' are analytic sentences as defined above. (His other paradigm examples, however, 'red is a color' and 'red is not a shape' are not analytic, as argued above).

Analytic sentences are *not* known by *pure thought* or *metaphysical reasons*. Instead, they are true in-a-language just in virtue of their fixed definiens terms and syntax. Such sentences are knowable as entailed 'truths-in-a-language.' *A priority* isn't involved in Mares' examples of analyticity.

Metaphysical and Mathematical Axioms are not *A Priori* Truths

Another class of alleged *a priori* truths are the axioms of metaphysical systems and formal deductive systems. This kind of alleged *a priori* truth is illustrated with the alleged necessary self-identity of Venus.

Our response here is that the sentence, 'If it exists, then planet Venus is identical to itself' is a substitution instance of the identity axiom: For any object **x**, it is necessarily the case that **x** is identical with **x**. With the substitution of any existing object (e.g., Venus) the sentence becomes a tautology. There is no need to appeal to anything other than the 'identity' axiom which allows the intelligible insertion of substitution instances for the variable **x**.

Similarly, sentences of the form 'either **p** or not-**p**' reflect the prescribed axioms of bivalence and excluded middle. The principles of 'bivalence' and 'excluded middle' are adopted as axioms for truth-functional deductive systems, but axioms (as implicit definitions) are not literally true, much less necessary 'truths of reason.'

Colored Extended Things: *A Priori* or Empirical?

Mare's example of 'Every colored thing is extended' is more problematic, because it is questionable whether this is true. For example, consider the flash of a colored laser beam through space. Is this a perception of an extended colored thing? Is a colored laser beam an extended thing? Also, suppose that a neuron-firing in our brain is colored red (as is our blood). Is a physical 'neuron-firing' a red extended 'thing' (or is it dispersed functional 'thing')? Examples of this sort seem to be empirical assertions and not knowable from an analysis (or recognition) of conceptual implications. The sentence 'Every colored thing is extended' appears to be contingent and could be false.

Bonjour's Examples of *A Priori* Knowledge

Laurence Bonjour is a leading advocate of the existence of *a priori* justification and knowledge. In his *A Defense of Pure Reason* (1998) he states that there are two distinguishable aspects to the classical conception of *a priori* justification: namely, that **p** is justified independently of any appeal to experience[10] and by its appeal to reason or pure thought alone (p. 7). He follows the lead of Kant and the overall tradition by stipulating that a proposition will count as being justified *a priori* as long as no appeal to experience is needed for the proposition to be justified (p. 10). He says the main thrust of the idea of *a priori* justification is of 'justification' that derives from pure thought or reason alone with no positive dependence on experience (p. 11). Besides epistemic concerns, Bonjour is also concerned with 'metaphysical necessity' where the status of a proposition is related to the ways the world might have been. He accepts the distinction between necessary, contingent, and necessary falsehoods, as described above. Bonjour's examples of *a priori* knowledge include the following:

(1) The axioms of deductive logic.
- The law of identity.

(2) The standard inference rules of deductive logic, and application to deductive arguments or inferences in reasoning, including transitivity inferences.
- Modus ponens.
- If one event is later than a second and the second is later than the third, then the first is later than the third.

[10] Bonjour (2011) counts 'experience' as kinesthetic experiences of body orientation in addition to the five senses, as well as introspective awareness of one's thoughts, sensations, and other mental states (p. 284). He says that there are certain foundational empirical beliefs, fully justified by appeal to direct experience or sensory observation alone (1998, p. 4).

(3) Propositions in logic and mathematics, including tautologies.

- 2 + 3 = 5.
- All cubes have 12 edges.
- For all **x**: **x** is the same number as **x**.

(4) Certain moral claims.

(5) The *a priori* is needed to account for justified inferences from directly experiential claims to further broadly empirical conclusions and for successful reasoning in general (a reply to Hume's problem).

(6) Common sense truths.

- Nothing can be both red and green all over at the same time.

(7) Alleged metaphysical truths e.g., about the structure of time and space.

- A physical object cannot be in two places at the same time.

Based upon the analysis of this chapter and previous chapters, each of Bonjour's examples of *a priori* knowledge is better explained by prescriptions and deductive fixed definiens necessities. It should be noted that the final two examples (6 & 7) appear to be contingent empirical truths. Let us consult our personal intuitions about 'common sense' truths and apparent truths about time and space:

(1) **S** knows that "red and green are two colors that *cannot* simultaneously appear (and coexist) on one physical body **x** at one time" because **S** has a sensory-based conception of these colors. Since these colors are opaque, it is physically "impossible for **x** to be painted both red and green." This is empirical.

(2) Similarly, **S** knows that "a physical item **x** *can* simultaneously appear (and coexist) as red and shellacked at the same time" if clear shellac (or varnish) is considered as a color. Since one color is transparent, it is "possible for **x** to be painted both red and with shellack at the same time." This is empirical.

(3) That a physical object cannot be in a single place at a single time is a contingent empirical truth as a causal law about solid extended bodies.

Bonjour's seven examples and attending arguments provide no evidence of an *a priori* mode of knowledge.[11]

V. Conclusion

In this chapter, it is argued that if, in fact, all so-called *a priori* knowledge is explainable as the knowledge of (1) analytic sentences (true-in-a-language based upon stipulated fixed definiens concepts), (2) theorems (entailed by relevant axioms, syntax, and inference rules), (3) tautologies (i.e., a sentence that is true for all possible truth-value assignments in its truth table) or (4) the *necessity* of the conclusion of a sound valid argument, then there is *no need for this extraneous epistemic concept*. Similar to the logical positivists, who wanted to reduce *a priori* knowledge to 'analytic' linguistic truths, I maintain that all *a priori* knowledge may be alternatively described as falling under any of these four categories of 'true-in-a-language' forms (i.e., as kinds of linguistic truths).

But again, if a theorist wants to steadfastly retain the concept of *a priori* knowledge as being defined as 'any knowledge that **p** without its justification based upon specific events in the empirical world,' then these four categories *can* be described as examples of *a priori* knowledge. It may be said that knowledge of 'truths-in-a-language' (e.g., analytic sentences, theorems, tautologies, deductive truths such as 2 + 2 = 4) can be said to be subject to *a priori* (non-empirical) *justificatory reasoning*, and as such are examples of *a priori knowledge*. But the worth of such an epistemic claim is questionable. Again, if *a priori* knowledge is only explainable, and disjunctively definable, as being based upon and justified by the vocabulary, axioms, formation rules, and inference rules (as *stipulations*) from natural and artificial languages, then the concept of *a priori* justification doesn't have much consequence (except for perhaps 'saving' a historically prominent term).

With respect to the concept of analyticity, the new definition of 'analyticity' proposed here, retains the historical core examples (e.g., 'all bachelors are unmarried,' 'a triangle has three sides'), as well as mathematical axioms (e.g., 'a straight line is a

[11] See Littlejohn and Carter, *This is Epistemology: An Introduction* (2021) pp. 98-105, for a concise description of arguments by Bonjour (1998) and George Bealer (1999a) that support *a priori* knowledge.

segment drawn between two points,' 'zero is a number') but the theoretical significance of this concept is limited. The concepts of '*a priori* knowledge' and 'analyticity,' while both theoretically retainable and historically influential, offer very little insight into the nature of the discipline of mathematics.[12]

[12] In a survey by Bourget and Chalmers (2023), it was found that 73% of philosophers believed in the existence of *a priori* knowledge, 18% denied its existence, with 9% answering 'other.' With respect to the existence of the analytic-synthetic distinction, 62% accept it, 26% deny it, and 12% answer 'other." From the perspective here, the 'other' option might be most appropriate to both issues. The reasoning here is that while the concepts of '*a priori* knowledge' and 'analyticity' might be afforded precise definitions that avoid inconsistency and cover some extensions, these definitions when fully articulated offer no insight into the nature of knowledge, mathematics, or language. In sum, these epistemic and semantic measurement concepts are not valuable. Analytic philosophy should avoid them.